$$I_2 = \frac{3}{8}\int_0^{2\pi} \sin \frac{n}{2} x\,dx = \frac{-3}{4n}\cos \frac{n}{2} x \Big]_0^{2\pi} = \begin{cases} 0 \\ \dfrac{6}{4n} \end{cases} \qquad n = even$$

$$I_3 = \frac{1}{2}\int_0^{2\pi} \cos 2x \sin \frac{n}{2} x\,dx = 0$$

F.S.S: $F(x) = \frac{3}{8} + \sum_{n=1}^{\infty} b_n \sin \frac{nx}{2}$

F.C.E: $F(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{ni\pi}{l}x}$ $\qquad 2\pi = 4 \Rightarrow l = \pi$

$$C_n = a_n + ib_n \begin{cases} C_2 = a_2 = \frac{1}{2} \\ C_4 = a_4 = \frac{1}{8} \\ C_0 = \frac{a_0}{2} = \frac{3}{8} \end{cases} \qquad , n \neq 2,4,0 \rightarrow C_n = 0$$

دامنه ی مرکب:

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{l}x-\beta_n\right)$$

$$tg^{-1}\downarrow\left(\frac{b_n}{a_n}\right)=\beta_n\,,A_n=\sqrt{a_n^2+b_n^2}$$

$$tg^{-1}\left(0/a_n\right)=0=\beta_n\,,A_n=a_n\Rightarrow\begin{cases}A_2=1/2\\ A_4=1/8\\ A_0=3/8\end{cases}$$

$$\cos^4 x=F(x)=3/8+1/2\cos 2x+1/8\cos 4x$$

قضیه دیریکله :

$x=\pi \longrightarrow$ $$F(x)=1=3/8+\frac{1}{2}\cos 2\pi+1/8\cos 4\pi$$

$1=3/8+\frac{1}{2}+\frac{1}{8}=1\Rightarrow 1=1$ درست

$x=0 \longrightarrow$ $F(0)=1=3/8+\frac{1}{2}\cos(0)+1/8\cos(0)=1$ درست

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32) سوال : $F(x)xe^x$ $\quad 0<x<\pi\Rightarrow 2l=\pi\Rightarrow l=\frac{\pi}{2}$

F.S: $$a_0=\frac{2}{\pi}\int_0^{\pi}xe^x dx=\frac{2}{\pi}\left(xe^x-e^x\right)\Big]_0^{\pi}=\frac{2}{\pi}\left(\pi e^{\pi}-e^{\pi}+e^0\right)$$

$$\Rightarrow a_0=\frac{2}{\pi}\left(\pi e^{\pi}-e^{\pi}+1\right)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} xe^x \cos 2nx dx$$

$$\int xe^x \cos 2nx dx = \frac{xe^x}{2n}\sin 2nx + \frac{e^x}{4n^2}\cos 2nx - \int \frac{xe^x}{2n}\sin 2nx dx + \int \frac{e^x}{4n^2}\cos 2nx]$$

$$, \int xe^x \sin 2nx dx \begin{cases} e^x = u \rightarrow du = e^x dx \\ dV = x\sin 2nx dx \Rightarrow V = -\frac{x}{2n}\cos 2nx + \frac{1}{4n^2}\sin 2nx \end{cases}$$

$$\Rightarrow \int xe^x \sin 2nx dx = -\frac{xe^x}{2n}\cos 2nx + \frac{1}{4n^2}\sin 2nx - \int -\frac{x}{2n}\cos 2nx e^x dx - \int \frac{1}{4n^2}\sin 2nx e^x dx, \ldots$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} xe^x \sin 2nx dx$$

F.S: $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2nx + b_n \sin 2nx$

F.C.S: $2\pi = 2l \Rightarrow l = \pi$

چون تابع زوج است. $b_n = 0$ $(\pi e^{\pi} - e^{\pi} + 1)$ همان

$a_n = \frac{1}{\pi} \times 2\int_0^{\pi} xe^x \cos nx dx$ ← به روش جز به جز بالا حل می شود.

F.C.S: $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

F.S.S: $b_n = \frac{1}{\pi} \times 2\int_0^{\pi} xe^x \sin nx$ به روش جز به جزو و

چون تابع فرد است. $a_n = 0$ دوبار

استفاده از آن بدست می آید.

F.S.S: $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin nx$

F.C.E: $F(x)=\sum_{n=-\infty}^{\infty} C_n e^{\frac{nin}{l}x}$

$$C_n = a_n + ib_n \quad \text{یا} \quad \begin{cases} C_n = \frac{1}{21}\int_{-l}^{l} F(x)e^{\frac{-ni\pi}{l}x}dx \\ C_0 = \frac{a_0}{2} \end{cases}$$

دامنه مرکب:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{l}x - \beta_n\right)$$

$$tg^{-1}\left(\frac{b_n}{a_n}\right) = \beta_n , A_n = \sqrt{a_n^2 + b_n^2}$$

$$\Rightarrow F(x) = \frac{1}{\pi}\left(\pi e^{\pi} - e^{\pi} + 1\right) + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n}{1}x - \beta_n\right)$$

اتحاد پارسوال:

$$\frac{1}{l}\int_{-l}^{l} F^2(x)dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\frac{2}{\pi}\int_0^{\pi}(xe^x)^2 dx = \frac{2}{\pi}\int_0^{\pi} x^2 e^{2x} dx = \frac{2}{\pi}\left(\frac{x^2}{2}e^{2x} - \frac{2x}{4}e^{2x} + \frac{1}{4}e^{2x}\right)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{2}e^{2\pi} - \frac{2\pi}{4}e^{2\pi} + \frac{1}{4}e^{2\pi} - \frac{1}{4}\right) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

سوال 33) $F(x) = \sin 2x \cos 4x \qquad 0 < x < \pi \qquad 2l = \pi \Rightarrow l = \frac{\pi}{2}$

F.S: $a_0 = \frac{2}{\pi}\int_0^{\pi} \sin 2x \cos 4x dx = \frac{1}{\pi}\int_0^{\pi} (\sin 6x - \sin 2x) dx$

$$= \frac{1}{\pi}\left(\frac{-1}{6}\cos 6x + \frac{1}{2}\cos 2x\right)\Big]_0^{\pi} = \frac{1}{\pi}\left(-\frac{1}{6} + \frac{1}{2} + \frac{1}{6} - \frac{1}{2}\right) = 0 \Rightarrow a_0 = 0$$

$$a_n = \frac{2}{2\pi}\int_0^{\pi} (\sin 6x - \sin 2x)\cos 2nx dx = \frac{1}{2\pi}\int_0^{\pi} (\sin(2n+6)x + \sin(6-2n)x) dx$$

$$- \frac{1}{2\pi}\int_0^{\pi} (\sin(2n+2)x + \sin(2-2n)x) dx$$

$$= \frac{1}{2\pi}\left(\frac{-\cos(2n+6)x}{2n+6} - \frac{\cos(6-2n)x}{6-2n}\right)\Big]_0^{\pi} - \frac{1}{2\pi}\left(\frac{-\cos(2n+2)x}{2n+2} - \frac{\cos(2-2n)x}{2-2n}\right)\Big]_0^{\pi} = 0$$

$\Rightarrow a_n = 0$

$$b_n = \frac{2}{2\times\pi}\int_0^{\pi} ((\sin 6x - \sin 2x)\sin 2nx) dx =$$

$$\frac{1}{2\pi}\int_0^{\pi} [\cos(6-2n)x - \cos(6+2x)x) - (\cos(2-2x)x - \cos(2+2n)x)]$$

$$= \frac{1}{2\pi}\left[\frac{\sin(6-2n)x}{6-2n} - \frac{\sin(6+2n)x}{6+2n} - \frac{\sin(2-2n)x}{2-2n} + \frac{\sin(2+2n)x}{2+2n}\right]_0^{\pi} = 0$$

$\Rightarrow b_n = 0$

F.S: $F(x) = \frac{1}{2}(\sin 6x - \sin 2x) = \frac{1}{2}\sin 6x - \frac{1}{2}\sin 2x$

$$\Rightarrow b_1 = -\frac{1}{2}, b_3 = \frac{1}{2}$$

F.C.S: $2\pi = 2l \Rightarrow \pi = l$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin 2n \cos 4x dx = 0 \Rightarrow a_0 = 0$$

$$a_n = \frac{1}{\pi} \times 2\int_0^{\pi} (\sin 2x \cos 4x) \cos nx dx = \frac{1}{\pi}\int_0^{\pi} (\sin 6x - \sin 2x) \cos nx dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} (\sin(6+n)x + \sin(6-n)x) dx - \frac{1}{2\pi}\int_0^{\pi} (\sin(2+n)x + \sin(2-n)x) dx$$

$$= \frac{1}{2\pi}\left(\frac{-\cos(6+n)}{6+n}x - \frac{\cos(-6+n)x}{6-n}\right)\Bigg]_0^{\pi} + \frac{1}{2\pi}\left(\frac{+\cos(2+n)x}{2+n} + \frac{\cos(2-n)x}{2-n}\right)\Bigg]_0^{\pi} = 0$$

$$a_n = \begin{cases} \frac{1}{2\pi}\left(\frac{2}{6+n} - 0 + \frac{2}{2+n} + 0\right) & n = odd \\ 0 & n = even \end{cases}$$

$b_n = 0$. چون تابع زوج است

F.C.S: $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

F.S.S: $a_0 = 0, a_n = 0$. چون تابع فرد است

$$b_n = \frac{1}{\pi} \times 2\left(\int_0^{\pi} (\sin 2x \cos 4x) \sin nx dx\right)$$

$$= \frac{1}{\pi}\int_0^{\pi} (\sin 6x - \sin 2x) \sin nx dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} (\cos(6-n)x - \cos(6+n)x) dx - \frac{1}{2\pi}\int_0^{\pi} (\cos(2-n)x - \cos(2+n)x) dx$$

$$= \frac{1}{2\pi}\left(\frac{\sin(6-n)x}{6-n} - \frac{\sin(6+n)x}{6+n}\right)\Bigg]_0^{\pi} - \frac{1}{2\pi}\left(\frac{\sin(2-n)x}{2-n} - \frac{\sin(n+2)}{2+n}x\right)\Bigg]_0^{\pi} = 0$$

$$\Rightarrow b_n = 0$$

F.S.S: $F(x) = \frac{1}{2}\sin 6x - \frac{1}{2}\sin 2x$

$$b_2 = -\frac{1}{2}, b_6 = \frac{1}{2}$$

F.C.E: $$F(x)=\sum_{n=-\infty}^{\infty} C_n e^{\frac{ni\pi}{l}x}, C_n=a_n+ib_n \to \begin{cases} C_1=+i\left(-\frac{1}{2}\right) \\ C_3=0+\frac{i}{2} \\ C_0=\frac{a_0}{2}=0 \end{cases} \Rightarrow F(x)=i\left(\frac{1}{2}-\frac{1}{2}\right)=0$$

دامنه ی مرکب:

$$F(x)=A_0+\sum_{n=1}^{\infty} A_n \cos(2nx-\beta_x)$$

$$tg^{-1}\left(\frac{b_n}{a_n}\right)=\beta_n, A_n=\sqrt{a_n^2+b_n^2} \begin{cases} A_1=b_1=+\frac{1}{2} \\ A_3=b_3=+1/2 \end{cases}$$

قضیه دیریکله:

$\xrightarrow{x=0}$ $F(0^+)=0 \Rightarrow F(x)=\frac{a_0}{2}+\sum_{n=0}^{\infty} a_n \cos 2nx + b_n \sin 2nx$

$F(0)=0=0$ درست $(0=0)$

بدیهی

$\xrightarrow{x=\pi}$ $F(\pi^-)=0 \Rightarrow F(\pi^+)=0=0$ درست $(0=0)$

بدیهی

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34) جواب

$$F(t)=\begin{cases} 1 & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$$

$y''-y=F(t) \qquad y(0)=0 \qquad y'(0)=0$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} 1dx = 1 \qquad a_n = \frac{1}{\pi}\int_0^{\pi} \cos nx = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin xdx = \begin{cases} 0 & n = even \\ \dfrac{2}{n\pi} & n = odd \end{cases}$$

$$F(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi}\sin(2n-1)t$$

$$\Rightarrow y''-y = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{2n-1}\sin(2n-1)t$$

$$y(t) = C_1e^t + C_2e^{-t}, y_{p1} = -\frac{1}{2}, y_{p2} = A_1\sin(2n-1)t + A_2\cos(2n-1)t$$

$$y''-y = \left((2n-1)^2-1\right)A_1\sin(2n-1)t - \left((2n-1)^2+1\right)A_2\cos(2n-1)t = 0$$

$$\frac{2}{n-1}\sin(2n-1)t = A_2 = 0, A_{1n} = \frac{2}{(2n-1)(1+(2n-1)^2)}$$

$$\Rightarrow y_{p2} = \sum_{n=1}^{\infty} \frac{-2}{(2n-1)(1+(2n-1)^2)}\sin(2n-1)t$$

$$\Rightarrow y(t) = C_1e^t + C_2e^{-t} - \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2\sin(2n-1)t}{(2n-1)(1+(2n-1)^2)}$$

$$y(0) = 0 \Rightarrow C_1 + C_2 = +\frac{1}{2}$$

$$y'(0) = 0 \Rightarrow C_1 - C_2 - \sum_{n=1}^{\infty} \frac{2}{1+(2n-1)^2} = 0 \begin{cases} C_1 = \dfrac{1}{4\pi} + \displaystyle\sum_{n=1}^{\infty} \frac{1}{1+(2n-1)^2} \\ C_2 = +\dfrac{1}{4\pi} - \displaystyle\sum_{n=1}^{\infty} \frac{1}{1+(2n-1)^2} \end{cases}$$

35)

$$F(t) = \begin{cases} 1 & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

$$y''+y = F(t) \qquad y(0) = y'(0) = 0$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} 1dt = 1$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos ntdt = \frac{\sin nt}{nt}\int_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin ntdt = -\frac{\cos nt}{nt}\Big|_0^{\pi} = \frac{\cos 0 - \cos nt}{n\pi} = \frac{1+(-1)^{n+1}}{n\pi}$$

$$b_n = \begin{cases} 0 & n \text{ زوج} \\ \dfrac{2}{n\pi} & n \text{ فـرد} \end{cases}$$

$$F(t) = \tfrac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)t$$

$$Y''+Y = F(t) \qquad D^2 + 1 = 0 \Rightarrow D = \pm i$$

$$Y_h = C_1 \cos t + C_2 \sin t$$

$$\begin{cases} Y_{p1} = K \\ Y''_{p1} = 0 \end{cases} \qquad K = \tfrac{1}{2} \qquad Y_{p1} = \tfrac{1}{2}$$

$$n = 1 \Rightarrow \sin t \Rightarrow Y_{p2} = (A_1 \sin t + A_2 \cos t)t$$

$$Y'_{p2} = (A_1 \cos t - A_2 \sin t)t + (A_1 \sin t + A_2 \cos t)$$

$$Y''_{p2} = (-A\sin t - A_2 \cos t)t + (A_1 \cos t - A_2 \sin t) + (A_1 \cos t - A_2 \sin t)$$

$$y''+y = 2(A_1 \cos t - A_2 \sin t) = \tfrac{1}{2} + \tfrac{2}{\pi}\sin t$$

$$A_1 = 0$$

$$-2A_2 = \tfrac{2}{\pi} \Rightarrow A_2 = -\tfrac{1}{\pi}$$

$$Y_{p3} = B_1 \sin(2n-1)t + B_2 \cos(2n-1)t$$

$$Y'_{p3} = (2n-1)B_1 \cos(2n-1)t - B_2(2n-1)\sin(2n-1)t$$

$$Y''_{p3} = -(2n-1)^2 B_1 \sin(2n-1)t - B_2(2n-1)^2 \cos(2n-1)t$$

$$Y''+Y = (B_1 \sin(2n-1)t + B_2 cods(2n-1)t)\{1-(2n-1)^2\} = \tfrac{1}{2} + \tfrac{2}{\pi}\sum_{n=2}^{\infty}\tfrac{1}{2n-1}\sin(2n-1)t$$

$$(1-(2n-1)^2)B_2 \cos(2n-1)t = 0 \qquad B_2 = 0$$

$$(1-(2n-1)^2)B_1\sin(2n-1)t = \frac{2}{\pi}\frac{1}{(2n-1)}\sin(2n-1)$$

$$B_{n1} = \frac{2}{\pi(2n-1)\left(1-(2n-1)^2\right)}$$

$$y_p = \frac{1}{2} + \left(-\frac{1}{\pi}\right)\sin t + \sum_{n=2}^{\infty}\frac{2}{\pi(2n-1)\left(1-(2n-1)^2\right)}$$

$$y(0)=0 \Rightarrow \frac{1}{2} + C_1 = 0 \Rightarrow C_1 = -\frac{1}{2}$$

$$y'(0)=0$$

$$y' = -\frac{1}{\pi}\cos t + \frac{2}{\pi(2n-1)^2\left(1-(2n-1)^2\right)}\sum_{n=2}^{\infty}\cos(2n-1)t - C_1\sin t + C_2\cos t$$

$$0 = -\frac{1}{\pi} + \frac{2}{\pi(2n-1)^2\left(1-(2n-1)^2\right)} + C_2 \Rightarrow C_2 = \frac{1}{\pi} - \frac{2}{\pi(2n-1)^2\left(1-(2n-1)^2\right)}$$

$$y_c = F(t) = \frac{1}{2} + \left(-\frac{1}{\pi}\right)\sin t\sum_{n=2}^{\infty}\frac{2}{\pi(2n-1)\left(1-(2n-1)^2\right)}\sin(2n-1)t$$

$$+\left(-\frac{1}{2}\right)\cos t + \left(\frac{1}{\pi} - \frac{2}{\pi(2n-1)^2\left(1-(2n-1)^2\right)}\right)\sin t$$

سوال 36

$$y''-y = F(t) = \begin{cases}1 & 0<t<\pi\\ 0 & \pi<t<2\pi\end{cases} \qquad y(0) = y''(0)$$

$$a_0 = \frac{1}{\pi}\left\{\int_0^{\pi}dt + \int_{\pi}^{2\pi}dt\right\} = 1$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}\cos nt\,dt = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin nt\,dt = \frac{1}{n\pi}\left(1-(-1)^n\right) = \begin{cases}n = even & 0\\ n = odd & \frac{2}{n\pi}\end{cases}$$

$$F(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{(2n-1)\pi}\sin(2n-1)t$$

y_{p1} $\qquad$ y_{p2}