$$y_p = y_{p1} + y_{p2} = \frac{\pi}{16} + y_{p2}$$

$$y_g = y_c + y_p = C_1 e^{-4t} + C_2 e^{-2t} + y_p \text{ (IV)}$$

$$y(0)=0 \qquad \to 0 = C_1 + C_2 + \frac{\pi}{16} + \sum_{n=1}^{\infty} \frac{2(8-n^2)((-1)^n - 1)}{\pi n^2 (n^4 + 20n^2 + 64)} \qquad \text{(II)}$$

$$y'(0) = 0$$

$$y'_g(0) = 0 \qquad \to y' = -4c_1 e^{-4t} - 2c_2 e^{-2t} + \sum_{n=1}^{\infty} \frac{12}{\pi} \frac{\left((-1)^n - 1\right)}{\left(n^4 + 20n^2 + 64\right)} \cos nt - \frac{2(8-n^2)((-1)^n - 1)}{\pi n (n^4 + 20n^2 + 64)} \sin nt$$

$$\to 0 = -4c_1 - 2c_2 + \sum_{n=1}^{\infty} \frac{12}{\pi} \frac{\left((-1)^n - 1\right)}{(n^4 + 20n^2 + 64)} \text{ (III)}$$

C_2, C_1 از معادلات (II) و (III) بدست می آیند.

پس از پیدا کردن C_2, C_1 مقادیر آنها را در معادله (IV) جاگذاری می کنیم.

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43) $F(x,y) = e^{x+y} \qquad , 0 < x < 2 \qquad , 0 < y < 2\pi$

$a = 1 \qquad , b = \pi$

$$F(x,y) = \frac{a_0(y)}{2} + \sum_{n=1}^{\infty} \left(a_m(y) \cos \frac{m\pi}{a} x + b_m(y) \sin \frac{m\pi}{a} x \right)$$

$$a_0(y) = \int_0^2 e^{x+y} dx = e^y (e^2 - 1)$$

$$a_m(y) = \int_0^2 e^{x+y} \cos m\pi x dx = e^y \int_0^2 e^x \cos m\pi x dx$$

$(+)e^x$	$\cos m\pi x$
$(-)e^x$	$\frac{1}{m\pi}\sin m\pi x$
$(+)e^x$	$-\frac{1}{m^2\pi^2}\cos m\pi x$
	$\int -\frac{1}{m^2\pi^2}\cos m\pi x$

$$a_m(y)=e^y\left[\frac{e^x}{m\pi}\sin m\pi x+\frac{e^x}{m^2\pi^2}\cos m\pi x-\frac{1}{m^2\pi^2}\int_0^2 e\cos m\pi x dx\right]_0^2$$

$$=e^y\int_0^2 e^x\cos m\pi x dx$$

$$\Rightarrow\left(1+\frac{1}{m^2\pi^2}\right)\int_0^2 e^x\cos m\pi x dx=\frac{1}{m^2\pi^2}(e^2-1)$$

$$\Rightarrow a_m(y)=\frac{e^y(e^2-1)}{m^2\pi^2+1}$$

$$b_m(y)=\int_0^2 e^{x+y}\sin m\pi x dx=e^y\int_0^2 e^x\sin m\pi x dx$$

$(+)e^x$	$\sin m\pi x$
$(-)e^x$	$\frac{-1}{m\pi}\cos m\pi x$
$(+)e^x$	$-\frac{1}{m^2\pi^2}\sin m\pi x$
	$\int -\frac{1}{m^2\pi^2}\sin m\pi x$

$$b_m(y)=e^y\int_0^2 e^x\sin m\pi x dx=e^y\left[\frac{-e^x}{m\pi}\cos m\pi x+\frac{e^x}{m^2\pi^2}\sin m\pi n\right.$$

$$\left.-\frac{1}{m^2\pi^2}\int_0^2 e^x\sin m\pi x dx\right]_0^2\Rightarrow$$

$$\left(1+\frac{1}{m^2\pi^2}\right)\int_0^2 e^x\sin m\pi x dx=\left(\frac{-1}{m\pi}\right)(e^2-1)\rightarrow$$

$$b_m(y)=\frac{m\pi e^y(1-e^2)}{1+m^2\pi^2}$$

$$F(x,y)=\frac{e^y(e^2-1)}{2}+\sum_{m=1}^{\infty}\left(\frac{e^y(e^2-1)}{m^2\pi^2+1}\cos m\pi x+\frac{m\pi e^y(1-e^2)}{1+m^2\pi^2}\sin m\pi x\right)\ \text{(I)}$$

$$e^{y} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n e \frac{n\pi}{b} y + b_n \sin\frac{n\pi}{b} y \right)$$

$$\rightarrow e^{y} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos ny + b_n \sin ny\right)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{y} dy = \frac{1}{\pi}(e^{2\pi} - 1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{y} \cos ny dy \Rightarrow$$

$(+)e^{y}$	$\cos ny$
$(-)e^{y}$	$\frac{1}{n}\sin ny$
$(+)e^{y}$	$-\frac{1}{n^2}\cos ny$
	$\int -\frac{1}{n^2}\cos ny$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{y} \cos ny dy = \frac{1}{\pi}\left[\left(\frac{e^{y}}{n}\sin ny + \frac{e^{y}}{n^2}\cos ny - \frac{1}{n^2}\int e^{y}\cos ny\right)\right]_0^{2\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}\frac{\left(e^{2\pi} - 1\right)}{n^2 + 1}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{y} \sin ny dy \Rightarrow$$

$(+)e^y$	$\sin ny$
$(-)e^y$	$-\frac{1}{n}\cos ny$
$(+)e^y$	$-\frac{1}{n^2}\sin ny$
	$\int -\frac{1}{n^2}\sin ny$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^y \sin ny dy = \frac{1}{\pi}\left[\left(\frac{-e^y}{n}\cos ny + \frac{e^y}{n^2}\sin ny - \frac{1}{n^2}\int e^y \sin ny\right)\right]_0^{2\pi}$$

$$\Rightarrow \int_0^{2\pi} e^y \sin ny dy = \frac{n(1-e^{2\pi})}{n^2+1} \rightarrow b_n = \frac{n(1-e^{2\pi})}{\pi(n^2+1)}$$

$$e^y = \frac{1}{2\pi}\left(e^{2\pi}-1\right) + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\frac{(e^{2\pi}-1)}{n+1}\cos ny + \frac{n}{\pi}\frac{\left(1-e^{2\pi}\right)}{\left(n^2+1\right)}\sin ny\right) \quad \text{(II)}$$

با جاگذاری عبارت (II) در طرف راست عبارت (I) سری فوریه دوگانه تابع بدست می آید.

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44) $F(x,y) = x^2 \sin 4y \qquad ,0 < x < 2\pi \qquad ,0 < y < 2$

$a = \pi \qquad ,b = 1$

$$F(x,y) = \frac{a_0(y)}{2} + \sum_{m=1}^{\infty}\left(a_m(y)\cos m\frac{\pi}{a}x + b_m(y)\sin\frac{m\pi}{a}x\right)$$

$$\rightarrow F(x,y) = \frac{a_0(y)}{2} + \sum_{m=1}^{\infty}\left(a_m(y)\cos mx + b_m(y)\sin mx\right)$$

$$a_0(y) = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin 4y dx = \frac{8}{3}\pi^2 \sin 4y$$

$$a_m(y) = \frac{\sin 4y}{\pi}\int_0^{2\pi} x^2 \cos mx dx \Rightarrow$$

$$a_m(y) = \frac{\sin 4y}{\pi}\left(\frac{x^2}{m}\sin mx + \frac{2n}{m^2}\cos mx - \frac{2}{m^3}\sin mx\right)_0^{2\pi} = \frac{4\sin 4y}{m^2}$$

$(+)x^2$	$\cos mx$
$(-)2x$	$\frac{1}{m}\sin mx$
$(+)2$	$\frac{-1}{m^2}\cos mx$
0	$\frac{-1}{m^3}\sin mx$

$$b_m(y) = \frac{\sin 4y}{\pi}\int_0^{2\pi} x^2 \sin mx dx \Rightarrow$$

$(+)x^2$	$\sin mx$
$(-)2x$	$\frac{-1}{m}\cos mx$
$(+)2$	$\frac{-1}{m^2}\sin mx$
0	$\frac{1}{m^3}\cos mx$

$$b_m(y) = \frac{\sin 4y}{\pi}\left(\frac{-x^2}{m}\cos mx + \frac{2n}{m^2}\sin mx + \frac{2}{m^3}\cos mx\right)_0^{2\pi} = \frac{\sin 4y}{\pi}\times\frac{-4\pi^2}{m} =$$

$$\frac{-4\pi}{m}\sin 4y \rightarrow b_m(y) = \frac{-4\pi}{m}\sin 4y$$

$$F(x,y) = \frac{4\pi^2}{3}\sin 4y + \sum_{m=1}^{\infty}\left(\frac{4\sin 4y}{m^2}\cos mx - \frac{4\pi}{m}\sin 4y \sin mx\right) \quad \text{(I)}$$

$$\sin 4y = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi y + b_n \sin n\pi y\right)$$

$$a_0 = \int_0^2 \sin 4y dy = \left(-\frac{1}{4}\cos 4y\right)_0^2 = \frac{1-\cos 8}{4}$$

$$a_n = \int_0^2 \sin 4y \cos n\pi y dy = \frac{1}{2}\int_0^2 \left[\sin(4y + n\pi y) + \sin(4y - n\pi y)\right]dy =$$

$$\frac{1}{2}\left[\frac{-1}{4+n\pi}\cos(4+n\pi)y - \frac{1}{4-n\pi}\cos(4-n\pi)y\right]_0^2 = \frac{4(1-\cos 8)}{16-n^2\pi^2}$$

$$b_n = \int_0^2 \sin 4y \sin n\pi y dy = \frac{1}{2}\int_0^2 \left[\cos(4y - n\pi y)dy - \cos(4y + n\pi y)\right]dy =$$

$$\frac{1}{2}\left[\frac{1}{4-n\pi}\sin(4-n\pi)y - \frac{1}{4+n\pi}\sin(4+n\pi)y\right]_0^2 = \frac{n\pi(\sin 8)}{16-n^2\pi^2}$$

$$\sin 4y = \frac{(1-\cos 8)}{8} + \sum_{n=1}^{\infty} \frac{4(1-\cos 8)}{16-n^2\pi^2}\cos n\pi y + \frac{n\pi}{16-n^2\pi^2}\sin 8 \sin n\pi y \quad (\mathrm{I})$$

با جاگذاری عبارت (II) در طرف راست عبارت (I) سری فوریه دوگانه تابع بدست می آید.

45) $F(x,y) = xy^2 + yx^3 \qquad ,0 < x < 1 \qquad ,0 < y < \pi$

$$a = \frac{1}{2} \qquad ,b = \frac{\pi}{2}$$

$$F(x,y) = \frac{a_0(y)}{2} + \sum_{m=1}^{\infty}\left(a_m(y)\cos 2m\pi x + b_m(y)\sin 2m\pi x\right)$$

$$a_0(y) = 2\int_0^1 (xy^2 + yx^3)dx = 2\left(\frac{x^2y^2}{2} + \frac{yx}{4}\right)_0^1 = 2\left(\frac{y^2}{2} + \frac{y}{4}\right) = \frac{2y^2+y}{2}$$

$$a_m(y) = 2\int_0^1 \left(xy^2 + yx^3\right)\cos 2m\pi x dx = \left[\int_0^1 2xy^2 2m\pi x dx + \int_0^1 2yx^3 \cos 2m\pi x dx\right]$$

I_1 $\qquad$ I_2

$$I_1 = \int_0^1 2xy^2 \cos 2m\pi x dx = y^2\left(\frac{x}{m\pi}\sin 2m\pi x + \frac{1}{2m^2\pi^2}\cos 2m\pi x\right)_0^1 = 0$$

$(+)2x$	$\cos 2m\pi x$
$(-)2$	$\frac{1}{2m\pi}\sin 2m\pi x$
0	$-\frac{1}{4m^2\pi^2}\cos 2m\pi x$

$$I_2 = \int_0^1 2yx^3 \cos 2m\pi x dx = 2y\left(\frac{x^3}{2m\pi}\sin 2m\pi x + \frac{3x^2}{4m^2\pi^2}\cos 2m\pi x - \frac{3x}{4m^3\pi^3}\sin 2m\pi x\right.$$

$(+)x^3 \qquad \cos 2m\pi x$

$(-)3x^2 \qquad \frac{1}{2m\pi}\sin 2m\pi x$

$(+)6x \qquad -\frac{1}{4m^2\pi^2}\cos 2m\pi x$

$(-)6 \qquad -\frac{1}{8m^3\pi^3}\sin 2m\pi x$

$0 \qquad \frac{1}{16m^4\pi^4}\cos 2m\pi x$

$$\left.-\frac{3}{8m^4\pi^4}\cos m\pi x\right)_0^1 = \frac{3y}{2m^2\pi^2}$$

$$\Rightarrow a_m(y) = \frac{3y}{2m^2\pi^2}$$

$$b_m(y) = 2\int_0^1 \left(xy^2 + yx^3\right)\sin 2m\pi x dx = \int_0^1 2xy^2 \sin 2m\pi x + \int_0^1 2yx^3 \sin 2m\pi x dx$$ b

I1 I2

$$I_1 = \int_0^1 2ny^2 \sin 2m\pi x = y^2\left(\frac{-x}{m\pi}\cos 2m\pi x + \frac{1}{2m^2\pi^2}\sin 2m\pi x\right)_0^1 = \frac{-y^2}{m\pi}$$

$(+)2x \qquad \sin 2m\pi x$

$-2 \qquad -\frac{1}{2m\pi}\cos 2m\pi x$

$0 \qquad -\frac{1}{4m^2\pi^2}\sin 2m\pi x$

$$I_2 = \int_0^1 2yx^3 \sin 2m\pi x dx = 2y\left(\frac{-x^3}{2m\pi}\cos 2m\pi x + \frac{3x}{4m^2\pi^2}\sin 2m\pi x + \frac{3}{4}\frac{x}{m^3\pi^3}\cos 2m\pi x\right.$$

$+x^3$	$\sin 2m\pi x$
$-3x^2$	$-\dfrac{1}{2m\pi}\cos 2m\pi x$
$+6x$	$-\dfrac{1}{4m^2\pi^2}\sin 2m\pi x$
-6	$\dfrac{1}{8m^3\pi^3}\cos 2m\pi x$
0	$\dfrac{1}{16m^4\pi^4}\sin 2m\pi x$

$$-\frac{3}{8m^4\pi^4}\sin 2m\pi x\Big)_0^1 = \left(\frac{3y}{2m^3\pi^3}-\frac{y}{m\pi}\right)$$

$$b_m(y) = I_1 + I_2 = \frac{3y}{2m^3\pi^3} - \frac{(y^2+y)}{m\pi}$$

$$F(x,y) = \frac{2y+y}{4} + \sum_{m=1}^{\infty}\left(\frac{3y}{2m^2\pi^2}\cos 2m\pi x + \left(\frac{3y}{2m^3\pi^3} - \frac{(y^2+y)}{m\pi}\right)\sin 2m\pi x\right) \text{(I)}$$

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{b}y + b_n \sin\frac{n\pi}{b}y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2ny + b_n \sin 2ny$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} y\,dy = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} y\cos 2ny\,dy = \frac{2}{\pi}\left(\frac{y}{2n}\sin 2ny + \frac{1}{4n^2}\cos ny\right)_0^{\pi} = 0$$

$+y$	$\cos 2ny$
-1	$\dfrac{1}{2n}\sin 2ny$
0	$-\dfrac{1}{4n^2}\cos 2ny$

$$b_n = \frac{2}{\pi}\int_0^{\pi} y\sin 2ny\,dy = \frac{2}{\pi}\left(\frac{-y}{2n}\cos 2ny + \frac{1}{4n^2}\sin 2ny\right)_0^{\pi} = \frac{-1}{n}$$

$+y$	$\sin 2ny$
-1	$\frac{-1}{2n}\cos 2ny$
0	$-\frac{1}{4n^2}\sin 2ny$

$$\Rightarrow y=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{-1}{n}\sin 2nydy$$

$$y^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos 2ny+b_n\sin 2ny$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}y^2dy=\frac{2\pi^2}{3} \qquad ,a_n=\frac{2}{\pi}\int_0^{\pi}y^2\cos 2nydy=$$

$+y^2$	$\cos 2ny$
$-2y$	$\frac{1}{2n}\sin 2ny$
$+2$	$-\frac{1}{4n^2}\cos 2ny$
0	$-\frac{1}{8n^3}\sin 2ny$

$$=\frac{2}{\pi}\left(\frac{y^2}{2n}\sin 2ny+\frac{y}{2n^2}\cos 2ny-\frac{1}{4n^3}\sin 2ny\right)_0^{\pi}=\frac{1}{n^2}$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}y^2\sin 2nydy=\frac{2}{\pi}\left[\frac{-y^2}{2n}\cos 2ny+\frac{y}{2n^2}\sin 2ny+\frac{1}{4n^3}\cos 2ny\right]_0^{\pi}=\frac{-\pi}{n}$$

$+y^2$	$\sin 2ny$
$-2y$	$-\frac{1}{2n}\cos 2ny$
$+2$	$-\frac{1}{4n^2}\sin 2ny$
0	$\frac{1}{8n^3}\cos 2ny$

$$\Rightarrow y^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2ny+\frac{-\pi}{n}\sin 2ny$$

اگر به جای y^2,y در طرف راست معادله (I) مقدارش را از عبارات بالا قرار دهیم سری فوریه دوگانه تابع بدست می آید.

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46) $F(x,y)=2y\sin(x-y)$ $\qquad 0<x<2\pi \qquad ,0<y<2$

$a=\pi \qquad ,b=1 \qquad F(x,y)=\frac{a_0(y)}{2}+\sum_{m=1}^{\infty}a_m(y)\cos\frac{m\pi}{a}x+b_m(y)\sin\frac{m\pi}{a}n$

$$F(x,y)=\frac{a_0(y)}{2}+\sum_{m=1}^{\infty}a_m(y)\cos mx+b_m(y)\sin mx$$

$$a_0(y)=\frac{1}{\pi}\int_0^{2\pi}2y\sin(x-y)dx=\left(\frac{-2y}{\pi}\cos(x-y)\right)_0^{2\pi}=0$$

$$a_m(y)=\frac{1}{\pi}\int_0^{2\pi}2y\sin(x-y)\cos mxdx=\frac{2y}{\pi}\int_0^{2\pi}\frac{1}{2}(\sin(x-y+mx)+\sin(x-y-mx))=$$

$$\frac{y}{\pi}\left[\frac{-1}{m+1}\cos(x-y+mn)+\frac{1}{(m-1)}\cos(x-y-mx)\right]_0^{2\pi}=0$$

$$b_m(y)=\frac{1}{\pi}\int_0^{2\pi}2y\sin(x-y)\sin mxdx=\frac{2y}{\pi}\int_0^{2\pi}\frac{1}{2}[\cos(x-y-mx)-\cos(x-y+mx)]$$

$$=\frac{y}{\pi}\left[\frac{1}{1-m}\sin(x-y-mx)-\frac{1}{1+m}\sin(x-y+mx)\right]_0^{2\pi}=0$$

سری فوریه دوگانه $\Rightarrow F(x,y)=2y[\sin x\cos y-\cos x\sin y]$

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47) $F(x,y)=\sinh x\cosh y$ $\qquad 0<x<1 \qquad ,0<y<1$

$a=\frac{1}{2} \qquad ,b=\frac{1}{2} \qquad F(x,y)=\frac{1}{2}(e^x-e^{-x})\times\frac{1}{2}(e^y+e^{-y})=$