$$F(x,y)+\frac{a_0(y)}{2}+\sum_{n=1}^{\infty}a_n(y)\cos nx+b_n(y)\sin nx$$

$$a_0(y)=\frac{1}{\pi}\int_0^{2\pi}(y^3x^2+xy)dx=\frac{8\pi^2}{3}y^3+2\pi y$$

$$a_a(y)=\frac{1}{\pi}\int_0^{2\pi}(y^3x^2+xy)\cos nxdx=\frac{y^3}{\pi}\int_0^{2\pi}x^2\cos nxdx+\frac{y}{\pi}\int_0^{2\pi}x\cos nxdx=\frac{4y^3}{n^2}+0=\frac{4y^3}{n^2}$$

$$b_n(y)=\frac{1}{\pi}\int_0^{2\pi}(y^3x^2+xy)\sin nxdx=\frac{y^3}{\pi}\int_0^{2\pi}x^2\sin nxdx+\frac{y}{\pi}\int_0^{2\pi}x\sin nxdx=\frac{-4\pi}{n}y^3-\frac{2y}{n}$$

$$F(x,y)=\frac{4\pi^2}{3}y^3+\pi y+\sum_{n=1}^{\infty}\frac{4y^3}{n^2}\cos nx+\left(-\frac{4\pi y^3}{n}-\frac{2y}{n}\right)\sin nx$$

$$y^3=2+\sum_{m=1}^{\infty}\frac{12}{m^2\pi^2}\cos m\pi y+\left(\frac{12}{m^3\pi^3}-\frac{8}{m\pi}\right)\sin m\pi y$$

$$y=1+\sum_{m=1}^{\infty}\left(\frac{-2}{m\pi}\right)\sin m\pi y$$

$$F(x,y)=\frac{4\pi^2}{3}\left\{2+\sum_{m=1}^{\infty}\frac{12}{m^2\pi^2}\cos m\pi y+\left(\frac{12}{m^3\pi^3}-\frac{8}{m\pi}\right)\sin m\pi y\right\}+\pi\left\{1+\sum_{m=1}^{\infty}\left(\frac{-2}{m\pi}\right)\sin m\pi y\right\}+$$

$$\sum_{n=1}^{\infty}\left[\frac{4\cos nx}{n^2}\left\{2+\sum_{m=1}^{\infty}\frac{12}{m^2\pi^2}\cos m\pi y+\left(\frac{12}{m^3\pi^3}-\frac{8}{m\pi}\right)\sin m\pi y\right\}+\right.$$

$$\left.\left\{-\frac{4\pi}{n}\left\{2+\sum_{m=1}^{\infty}\frac{12}{m^2\pi^2}\cos m\pi y+\left(\frac{12}{m^3\pi^3}-\frac{8}{m\pi}\right)\sin m\pi y\right\}-\frac{2}{n}\left\{1+\sum_{m=1}^{\infty}\left(\frac{-2}{m\pi}\right)\sin m\pi y\right\}\right\}\sin nx\right]$$

$$F(x)=\begin{cases}0 & -\infty<x<0\\ \sin\frac{\pi}{4}x & 0<x<4\\ 0 & 4<x<\infty\end{cases}\qquad F(w)=\int_0^4\sin\frac{\pi x}{4}e^{-iwx}dx=\frac{\frac{4}{\pi}\left(e^{-4iw}+1\right)}{\left(1-\frac{16}{\pi^2}w^2\right)}$$

D I

$(+)e^{-iwx}$	$\sin\frac{\pi x}{4}$
$(-)-iwe^{-iwx}$	$-\frac{4}{\pi}\cos\frac{\pi}{4}x$
$(+)i^2w^2e^{-iwx}$	$-\frac{16}{\pi^2}\sin\frac{\pi}{4}x$
	$\int -\frac{16}{\pi^2}\sin\frac{\pi}{4}x$

$$I=\frac{-4}{\pi}e^{-iwx}\cos\frac{\pi}{4}x-\frac{16}{\pi^2}iwe\sin^{-iwx}\frac{\pi}{4}x$$
$$+w^2\frac{16}{\pi^2 I}$$

$$I\left(1-\frac{16}{\pi^2}w^2\right)=\left\{-\frac{4}{\pi}e^{-iwx}\cos\frac{\pi}{4}x-\frac{16}{\pi^2}iwe^{-iwx}\sin\frac{\pi}{4}x\right\}_0^4=\frac{4}{\pi}e^{-4iw}+\frac{4}{\pi}=\frac{4}{\pi}(e^{-4iw}+1)$$

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)e^{-wx}dw=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\frac{4}{\pi}(e^{-4iw}+1)}{\left(1-\frac{16}{\pi^2}w^2\right)}e^{iwx}dw=2\int_{-\infty}^{\infty}\frac{e^{-4iw}+1}{\pi^2-16w^2}e^{-wx}dw$$

ج) 53

$$\int_{-\infty}^{\infty}\frac{e^{-4iw}+1}{\pi^2-16w^2}dw=0$$

$x=0\rightarrow$ ناپیوسته است

$$\Rightarrow\int_{-\infty}^{\infty}\frac{e^{iw(x-4)}+e^{iwx}}{\pi^2-16w^2}\uparrow$$

$x=4\rightarrow$ تابع پیوسته است.

$$\Rightarrow\int_{-\infty}^{\infty}\frac{1+e^{+iw}}{\pi^2-16w^2}dw=0$$

هـ) 54

$$F(x)=\begin{cases}2-x & 0<x\le 2\\ 0 & elsewhere\end{cases}$$

$$F(w)=\int_0^2(2-x)e^{-iwx}dx=\left\{\frac{-(2-x)}{iw}e^{-iwx}-\frac{1}{w^2}e^{-iwx}\right\}_0^2$$

$$=-\frac{1}{w^2}e^{-2iw}+\frac{2}{iw}+\frac{1}{w^2}=\frac{2}{iw}+\frac{1}{w^2}(1-e^{-2iw})$$

D	I
$(+)2-x$	e^{-iwx}
$(-)-1$	$-\frac{1}{iw}e^{-iwx}$
$(+)0$	$\frac{1}{i^2w^2}e^{-iwx}$

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2}{iw}+\frac{1}{w^2}(1-e^{-2iw})\right]e^{iwx}dw$$

ج) 54)

$x=0\to$ تابع ناپیوسته است $\to\frac{0+2}{2}=1$

$$\Rightarrow 2\pi=\int_{-\infty}^{\infty}\left[\frac{2}{iw}+\frac{1}{w^2}(1-e^{-2iw})\right]dw$$

$x=2\to$ تابع پیوسته است $\to 0$

$$\int_{-\infty}^{\infty}\left[\frac{2}{iw}+\frac{1}{w^2}(1-e^{-2iw})\right]dw=0$$

هـ) 55)

$$F(x)=\begin{cases}1-x^2 & -1<x\le 0\\ \cos x & 0<x\le \dfrac{\pi}{2}\\ 0 & elsewhere\end{cases}\qquad F(w)=\int_{-1}^{0}(1-x^2)e^{-iwx}dx+\int_{0}^{\frac{\pi}{2}}\cos xe^{-iwx}dx=$$

$$\int_{-1}^{0}(1-x^2)e^{-iwx}dx=-\frac{1}{iw}+\frac{2}{iw^3}(e^{iw}-1)-\frac{2}{w^2}e^{iw}$$

$$\int_{0}^{\frac{\pi}{2}}\cos xe^{-iwx}dx=\frac{e^{-\frac{iw\pi}{2}}+iw}{1-w^2}$$

$$F(w)=\frac{2}{iw^3}(e^{iw}-1)-\frac{2}{w^2}e^{iw}-\frac{1}{iw}+\frac{e^{-\frac{iw\pi}{2}}+iw}{1-w^2}$$

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)e^{iwx}dw=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{2}{iw^3}(e^{iw(x+1)}-e^{iwx})-\frac{2}{w}e^{iw(x+1)}-\frac{e^{iwx}}{iw}+\frac{e^{-\frac{iw\pi}{2}}+iw}{1-w^2}e^{iwx}\right\}dw$$

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55 ج)

$x=-1\rightarrow$ تابع پیوسته است.

$$\int_{-\infty}^{\infty}\left\{\frac{2}{iw^3}(1-e^{-iw})-\frac{2}{w^2}-\frac{e^{-iw}}{iw}+\frac{e^{-\frac{iw\pi}{2}}+iw}{1-w^2}e^{-iw}\right\}dw=0$$

$x=0\rightarrow$ تابع پیوسته است.

$$\int_{-\infty}^{\infty}\left\{\frac{2}{iw^3}(e^{iw}-1)-\frac{2}{w^2}e^{iw}-\frac{1}{iw}+\frac{e^{\frac{-iw\pi}{2}}+iw}{1-w^2}\right\}dw=1$$

هـ)

56) $$F(x)=\begin{cases}0 & -\infty<x<0\\ 2^{-x} & 0<x<\infty\end{cases}$$

$$F(w)=\int_{-\infty}^{\infty}F(x)e^{-iwx}dx \qquad ,F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)e^{iwx}dw$$

$$F(w)=\int_{0}^{\infty}2^{-x}e^{-iwx}dx \qquad \rightarrow\begin{cases}e^{-iwx}=u\rightarrow(-iw)e^{-iwx}dx=du\\ 2^{-x}dx=dV\rightarrow\left(\frac{1}{2}\right)xdx=dV\rightarrow V\end{cases}=\frac{\left(\frac{1}{2}\right)x}{\ln\frac{1}{2}}$$

$$F(w)=\int_{0}^{\infty}2^{-x}e^{-iwx}dx=\frac{2^{-x}e^{-iwx}}{\ln\frac{1}{2}}-\int\frac{2^{-x}}{-\ln 2}(-iw)e^{-iwx}dx=$$

$$\frac{2^{-x}e^{-iwx}}{-\ln 2}-\frac{iw}{\ln 2}\int_{0}^{\infty}2^{-x}e^{-iwx}dx\Rightarrow\left(1+\frac{iw}{\ln 2}\right)\int_{0}^{\infty}2^{-x}e^{-iwx}dx=\frac{2^{-x}e^{-iwx}}{-\ln 2}$$

$$\rightarrow\int_{0}^{\infty}2^{-x}e^{-iwx}dx=\left(\frac{-\left(2^{-x}e^{-iwx}\right)}{\ln 2+iw}\right)_{0}^{\infty}\rightarrow F(w)=\frac{1}{(\ln 2)+iw}$$

$$\rightarrow F(x)=\frac{1}{2\pi}\int_{0}^{\infty}\frac{1}{(\ln 2)+iw}e^{iwx}dw=2^{-x}$$

ح)

بـررسی قـضیـه دیـریـکـلـه $\Rightarrow x=0\Rightarrow\frac{1}{2\pi}\int_{0}^{\infty}\frac{1}{(\ln 2)+iw}dw=1\rightarrow$

$$\int_{0}^{\infty}\frac{1}{(\ln 2)+iw}dw=2\pi$$

هـ) سوال 57)

$$F(x)=\begin{cases}1+x & -1<x<0\\ 1 & 0<x<1\\ 0 & elsewhere\end{cases}$$

$$F(w)=\int_{-\infty}^{\infty}F(x)e^{-iwx}dx=\int_{-1}^{0}(1+x)e^{-iwx}dx+\int_{0}^{1}e^{-iwx}dx=-\frac{1}{iw}e^{-iwx}\Big|_{-1}^{0}$$

$$+\left[-\frac{e}{iw}e^{-iwx}+\frac{1}{w}e^{-iwx}\right]\Big|_{-1}^{0}-\frac{1}{iw}e^{-iwx}\Big|_{0}^{1}=-\frac{1}{iw}+\frac{1}{iw}e^{iw}-\frac{1}{iw}e^{iw}+\frac{1}{w^2}-\frac{1}{w^2}e^{iw}-\frac{1}{iw}e^{-iw}+\frac{1}{iw}=$$

$$\frac{1}{w^2}(1-e^{iw})-\frac{1}{iw}e^{iw}$$

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)e^{iwx}dw=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{w^2}(1-e^{iw})-\frac{1}{iw}e^{-iw}\right]e^{iwx}dw$$

ح) بررسی قضیه دیریکله :

$x=0$ پیوسته

$$F(0)=1=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{w^2}(1-e^{iw})-\frac{1}{iw}e^{-iw}\right]e^{\circ}dw$$

$x=1$ ناپیوسته

$$\frac{F(1^+)+F(1^-)}{2}=\frac{1}{2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{w^2}(1-e^{iw})-\frac{1}{iw}e^{-iw}\right]e^{iw}dw$$

سوال 58) قسمت هـ)

$$F(x)=x^3e^{-x2}\qquad -\infty<x<\infty$$

$$F(w)=\int_{-\infty}^{\infty}F(x)e^{-iwx}dx=\int_{-\infty}^{\infty}x^3e^{-x2}e^{-iwx}dx=\int_{-\infty}^{\infty}x^3e^{-(x^2+iwx)}dx$$

$$=\int_{-\infty}^{\infty}x^3e^{-\left[\left(x+\frac{iw}{2}\right)^2+\frac{w^2}{4}\right]}dx=e^{w2/4}\int_{-\infty}^{\infty}x^3e^{-(x+iw/2)^2}dx$$

$$*\quad x+iw/2=\lambda\rightarrow x=\lambda-iw/2\qquad ,dx=d\lambda$$

$$I=e^{-w2/4}\int_{-\infty}^{\infty}\left(\lambda-iw/2\right)^3e^{-\lambda^2}d\lambda=e^{-\frac{w^2}{4}}\int_{-\infty}^{\infty}\left[\lambda^3-\frac{3iw}{2}\lambda^2-\frac{3w^2\lambda}{4}+\frac{iw^3}{8}\right]e^{-\lambda^2}d\lambda$$

D	I
$(+)\lambda^2$	$\lambda e^{-\lambda^2}$
$(-)2\lambda$	$-\tfrac{1}{2}e^{-\lambda^2}$

$$\int \lambda e^{-\lambda^2}d\lambda$$

$$\int_{-\infty}^{\infty}\lambda^3 e^{-\lambda^2} = -\tfrac{1}{2}\lambda^2 e^{-\lambda^2} + \int_{-\infty}^{\infty}\lambda e^{-\lambda^2}d\lambda$$

$$= -\tfrac{1}{2}e^{-\lambda^2}\left(\lambda^2+1\right)\Big|_{-\infty}^{\infty}$$

$$\int_{-\infty}^{\infty}\lambda^2 e^{-\lambda^2}d\lambda = \lambda e^{-\lambda^2}\Big|_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty}e^{-\lambda^2}d\lambda$$

$$\int_{-\infty}^{\infty}\lambda e^{\lambda^2}d\lambda = -\tfrac{1}{2}e^{-\lambda^2}\Big|_{-\infty}^{\infty}$$

D	I
λ	$\lambda e^{-\lambda^2}$
1	$-\tfrac{1}{2}e^{-\lambda^2}$
	$+\tfrac{1}{2}\int e^{-\lambda^2}$

جایگذاری:

$$I = e^{-w^2/4}\left[-\tfrac{1}{2}e^{-\lambda^2}\left(\lambda^2+1\right)\Big|_{-\infty}^{\infty} - \frac{3iw}{2}\lambda e^{-\lambda^2}\Big|_{-\infty}^{\infty} - \frac{3iw}{4}\int_{-\infty}^{\infty}e^{-\lambda^2}d\lambda + \frac{3w^2}{8}e^{-\lambda^2}\Big|_{-\infty}^{\infty} + \frac{iw^3}{8}\int_{-\infty}^{\infty}e^{-\lambda^2}d\lambda\right]$$

$$= e^{-w^2/4}\left(\frac{iw^3}{8} - \frac{3iw}{4}\right)\int_{-\infty}^{\infty}e^{-\lambda^2}d\lambda = \sqrt{\pi}\,e^{-w^2/4}\left(\frac{iw^3}{8} - \frac{3iw}{4}\right)$$

$$* \int_{-\infty}^{\infty}e^{-t^2}dt = \sqrt{\pi}$$

انتگرال کلوین:

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)e^{iwx}dw = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\frac{w^2}{4}}\left(\frac{iw^3}{8} - \frac{3iw}{4}\right)e^{iwx}dw$$

بررسی قضیه دیریکله : (قسمت ح)

$x=0$ پیوسته

$F(0)=0$

$$F(0)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-w^2/4}\left(\frac{iw^3}{8}-\frac{3iw}{4}\right)dw=0$$

$x=1$ پیوسته

$F(1)=e^{-1}$

$$F(1)=e^{-1}=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-w^2/4}\left(\frac{iw^3}{8}-\frac{3iw}{4}\right)e^{iw}dw$$

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59) سوال : هـ

$$F(x)=\frac{2x}{(x^2+1)^2}\qquad 0<x<\infty$$

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)e^{iwx}dw\qquad F(w)=F\left(\frac{2x}{(1+x^2)^2}\right)=?$$

ابتدا $F\left(\frac{1}{1+x^2}\right)$ را حساب می کنیم :

می دانیم :

$$2\pi F(-w)=F(F(x))$$

$$F(-w)=\frac{1}{1+w^2}\Rightarrow 2\pi F(-w)=\frac{2\pi}{1+w^2}$$

تبدیل فوریه وارون تابع فوق را حساب می کنیم :

$$F^{-1}\left(\frac{2\pi}{1+w^2}\right)=2\pi\frac{e^{-|x|}}{2}$$

زیرا $\longrightarrow$ $F^{-1}\left(\frac{1}{a^2+w^2}\right)=\frac{e^{-a|x|}}{2a}\xrightarrow{\quad} a=1\qquad F^{-1}\left(\frac{1}{1+w^2}\right)=\frac{e^{-|x|}}{2}$

$$\Rightarrow F^{-1}\left(\frac{2\pi}{1+w}\right)=\pi e^{-|x|}=F(x)$$

$\rightarrow F(w)=\pi e^{-|x|} \rightarrow \frac{1}{1+x^2}$ تبدیل فوریه برای تابع

برزای یافتن تبدیل فوریه تابع $\frac{2x}{(1+x^2)^2}$ با توجه به

اینکه $-\left(\frac{1}{1+x^2}\right)'=\frac{2x}{(1+x^2)^2}$

پس داریم:

$$F\left(\frac{1}{1+x^2}\right)=\pi e^{-|w|}$$

$$\Rightarrow F\left(\frac{2x}{(1+x^2)^2}\right)=-iw\pi e^{-|w|}=F(w)$$

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}-iw\pi e^{-|w|}e^{iwx}dw$$

ح : قضیه دیریکله: $x=1 \rightarrow F(1)=\frac{2(1)}{(1+1)^2}=\frac{1}{2}$

$$\Rightarrow \frac{1}{2}=\frac{1}{2}\int_{-\infty}^{\infty}-iwe^{-|w|}e^{iw}dw \Rightarrow \int_{-\infty}^{\infty}iwe^{iw-|w|}dw=-1$$

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60)

$$F(x)=\frac{2}{x^2+4x+8}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(t)e^{iw//9x-t)}d+dw$$

$$I=\int_{-\infty}^{\infty}\frac{2}{(t+2)^2+4}e^{iw(x-t)}dt \quad , t+2=\lambda \rightarrow dt=d\lambda , t=$$

$$I=2e^{iwx}\int_{-\infty}^{\infty}\frac{e^{-iwt}}{(t+2)^2+4}dt=2e^{iw(x+2)}\int_{-\infty}^{\infty}\frac{e^{-iw\lambda}}{\lambda^2+4}d\lambda$$ (کتاب 771 ص)

$$= e^{iw(x+2)}\int_0^{\infty}\frac{e^{-iw\lambda}}{\lambda^2+4}d\lambda = e^{iw(x+2)}\int_0^{\infty}\frac{\cos w\lambda}{\lambda^2+4}d\lambda = e^{iw(x+2)}\times(\ldots)$$

$$= e^{iw(x+2)}\times\frac{\pi}{4}e^{-2w} \to \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi}{4}e^{iw(x+2)}\times e^{-2w}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi}{4}e^{w(ix+2i-2)}dw = \frac{1}{8}\left(\frac{1}{i(x+2)-2}e^{w(i(x+2)-2)}\Big|_{-\infty}^{\infty}\right)$$

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61)

$$F(x)=\frac{e^{4ix}}{x^2-6x+13}=\frac{e^{4ix}}{(x-3)^2+4}$$

با توجه به سوال قبل:

$$I=\int_{-\infty}^{\infty}\frac{e^{4it}\times e^{iw(x-t)}}{(t-3)^2+4}dt = e^{iwx}\int_{-\infty}^{\infty}\frac{e^{it(4-w)}}{(t-3)^2+4}$$

$$t-3=\lambda \to dt=d\lambda, t=\lambda+3 \to e^{iwx}\int_{-\infty}^{\infty}\frac{e^{i(4\lambda+12-w\lambda-3w)}}{\lambda^2+4}d\lambda = I$$

$$I=e^{iwx}e^{12i}e^{-3iw}\int_{-\infty}^{\infty}\frac{e^{i\lambda(4-w)}}{\lambda^2+4}d\lambda = 2e^{iw(x-3)+12i}\int_0^{\infty}\frac{\cos\lambda(4-w)}{\lambda^2+4}d\lambda$$

$$= 2e^{\cdots}\times\frac{\pi}{4}e^{-2(4-w)} \to \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi}{2}\times 2\times e^{\cdot\cdot}e^{-2(4-w)}dw$$

$$= \frac{1}{4}\int_{-\infty}^{\infty}e^{iw(x-3)+12i}e^{-2(4-w)}dw = \frac{1}{4}e^{12i}\times e^{-8}\int_{-\infty}^{\infty}e^{iw(x-3)}\times e^{2w}dw$$
