به نام خدا

دانشگاه تبریز
دانشکده مهندسی برق و کامپیوتر
گروه مهندسی الکترونیک

نمونه سوالات ریاضیات مهندسی

✓ معادلات دیفرانسیل پاره ای

2007

سوال (1)

$$C^2 u_{xx} = utt$$

$$B.C\begin{cases} u(0,t)=0 \\ u(l,t)=0 \end{cases}, \quad I.C\begin{cases} u(x,0)=F(x) \\ u_t(x,0)=g(x) \end{cases}$$

$$u(x,t)=M(x)N(t) \Rightarrow C^2 M^u N = N^u M$$

$$\Rightarrow \frac{M^2}{M} = \frac{N^u}{C^2 N} = \lambda = -\beta^2$$

$$\Rightarrow \begin{cases} M''+\beta^2 M = 0 \Rightarrow M(x) = A_1 \sin(\beta x) + A_2 \cos(\beta x) & (I) \\ N''+\beta^2 C^2 N = 0 \Rightarrow N(t) = B_1 \sin(\beta ct) + B_2 \cos(\beta ct) & (II) \end{cases}$$

$$: I : M(x) = A\sin(\beta x) + A\cos(\beta x)$$

$$B.C : \begin{cases} u(0,t)=0 \Rightarrow A_2 = 0 \Rightarrow A_1 \neq 0 \\ u(l,t)=0 \Rightarrow M(l)=0 \Rightarrow A_1 din(\beta l) = 0 \Rightarrow \beta_n = \frac{n\pi}{l}, n=1,2,3,\ldots \end{cases}$$

$$: II : N(t) = B_1 \sin(\beta ct) + B_2 \cos(\beta ct) \Rightarrow N(t) = B_1 \sin\left(\frac{nc\pi}{l} t\right) + B_2 \cos\left(\frac{nc\pi}{l} t\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l} x\right)\left[A_1 B_2 \cos\left(\frac{nc\pi}{l} t\right) + A_1 B_1 \sin\left(\frac{n\pi c}{l} t\right)\right]$$

$$: A_1 B_2 = A_n n, A_1 B_1 + B_n$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l} x\right)\left[A_n \cos\left(\frac{nc\pi}{l} t\right) + B_n \sin\left(\frac{n\pi c}{l} t\right)\right]$$

$$I.C : \begin{cases} u(x,0) = F(x) \rightarrow F(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l} x\right) & (III) \\ u_t(x,0) = g(x) \rightarrow g(x) = \sum_{n=1}^{\infty} \left(\frac{l}{nc\pi}\right) B_n \sin\left(\frac{n\pi}{l} x\right) & (IV) \end{cases}$$

$$: II : A_n = \frac{2}{l}\int_0^l F(x).\sin\left(\frac{n\pi}{l} x\right) dx$$

$$: IV : B_n = \frac{2}{cn\pi}\int_0^l g(x).\sin\left(\frac{n\pi}{l} x\right) dx$$

الف: $\begin{cases} F(x) = e^{-x} \\ g(x) = 0 \end{cases}$

چون: $g(x) = 0 \rightarrow B_n = 0$

$$: A_n = \frac{2}{l}\int_0^l e^{-x}.\sin\left(\frac{n\pi}{l}x\right)dx = \frac{2}{l}\left[\left(\frac{-e^{-x}}{\frac{n\pi}{l}}\cos\left(\frac{n\pi}{l}x\right) - \frac{e^{-x}}{\left(\frac{n\pi}{l}\right)^2}\sin\left(\frac{n\pi}{l}x\right)\right)\Bigg|_0^l - \int_0^l \frac{e^{-x}}{\left(\frac{n\pi}{l}\right)^2}.\sin\left(\frac{n\pi}{l}x\right)dx\right]$$

$$\Rightarrow \left(1 + \frac{l}{n^2\pi^2}\right) \times \frac{2}{l}\int_0^l e^{-x}\sin\left(\frac{n\pi}{l}x\right)dx = \frac{2}{l}\left[\frac{-le^{-x}}{n\pi}\cos\left(\frac{n\pi}{l}x\right) - \frac{l^2e^{-x}}{n^2\pi^2}\sin\left(\frac{n\pi}{l}x\right)\right]_0^l$$

$$\Rightarrow A_n = \frac{2}{l}\int_0^l e^{-x}.\sin\left(\frac{n\pi}{l}x\right) = \frac{2n\pi}{n^2\pi^2 + l^2}\left((-1)^{n+1} \times e^{-l} + 1\right)$$

ب: $\begin{cases} F(x) = 0 \\ g(x) = 48n\left(\frac{5\pi}{l}x\right) \end{cases}$

$$: F(x) = 0 \rightarrow A_n = 0$$

$$B_n = \frac{2}{nc\pi}\int_0^l \sin\left(\frac{n\pi}{l}x\right) \times 4\sin\left(\frac{5\pi}{l}x\right)dx = \frac{4}{nc\pi}\int_0^l \left[\cos(5-x)\frac{\pi}{l}x - \cos(5+n)\frac{\pi}{l}x\right]dx$$

$$= \frac{4}{nc\pi}\left[\frac{l.\sin(5-n)\frac{\pi}{l}x}{(5-n)\pi} - \frac{l.\sin(5+n)\frac{\pi}{l}x}{(5+n)\pi}\right]_0^l = 0, n \neq 5$$

$$: n = 5 \Rightarrow B = \frac{4}{5c\pi}\int_0^l \left[\cos(5-5)\frac{\pi}{l}x - \cos(5+5)\frac{\pi}{l}x\right]dx$$

$$= \frac{4}{5c\pi}\left[x - \frac{l.\sin\left(\frac{10\pi}{l}x\right)}{10\pi}\right]_0^l = \frac{4l}{5c\pi}$$

$$\Rightarrow B_5 = \frac{4l}{5c\pi}$$

ج: $\begin{cases} F(x) = \cos(\beta x) \\ g(x) = \sin(\beta x) \end{cases}$

$$: A_n = \frac{2}{l}\int_0^l \cos(\beta x).\sin\left(\frac{n\pi}{l}x\right)dx = \frac{1}{l}\int_0^l \left[\sin\left(\beta+\frac{n\pi}{l}\right)x + \sin\left(\beta-\frac{n\pi}{l}\right)x\right]dx$$

$$= \frac{1}{l}\left[\frac{-l.\cos\left(\beta+\frac{n\pi}{l}\right)x}{l\beta+n\pi} - \frac{l.\cos\left(\beta-\frac{n\pi}{l}\right)x}{l\beta-n\pi}\right]_0^l = \frac{1-\cos(\beta l+n\pi)}{\beta l+n\pi} + \frac{1-\cos(\beta l-n\pi)}{\beta l-n\pi}$$

$$: B_n = \frac{2}{nc\pi}\int_0^l \sin(\beta x)\sin\left(\frac{n\pi}{l}x\right)dx = \frac{1}{nc\pi}\int_0^l \left[\cos\left(\beta-\frac{n\pi}{l}\right)x - \cos\left(\beta+\frac{n\pi}{l}\right)\right]dx$$

$$= \frac{1}{nc\pi}\left[\frac{l.\sin\left(\beta-\frac{n\pi}{l}\right)x}{\beta l-n\pi} - \frac{l.\sin\left(\beta+\frac{n\pi}{l}\right)x}{\beta l+n\pi}\right]_0^l = \frac{l}{nc\pi}\left[\frac{\sin(\beta l-n\pi)}{\beta l-n\pi} - \frac{\sin(\beta l+n\pi)}{\beta l+n\pi}\right]$$

د: $\begin{cases} F(x) = x^2 e^{-x} \\ g(x) = 0 \end{cases}$

$$: g(x) = 0 \rightarrow B_n = 0$$

$$A_n = \frac{2}{l}\int_0^l x^2.e^{-x}.\sin\left(\frac{n\pi}{l}x\right)dx = \frac{2}{l}\int_0^l \frac{x^2}{2i}e^{-x}\left(e^{\frac{n\pi}{l}ix} - e^{-\frac{n\pi}{l}ix}\right)dx$$

$$= \frac{1}{il}\int_0^l \left[x^2.e^{\left(\frac{n\pi}{l}i-1\right)x} - x^2.e^{-\left(\frac{n\pi}{l}i+1\right)x}\right]dx$$

$$= \frac{1}{il}\left[\left(\frac{lx^2}{ni\pi-l} - \frac{2l^2x}{(ni\pi-l)^2} - \frac{2l^3}{(ni\pi)^3}\right)e^{\left(\frac{ni\pi}{l}-1\right)x} + \left(\frac{lx^2}{ni\pi+l} + \frac{2l^2x}{(ni\pi+l)} + \frac{2l^3}{(ni\pi+l)^3}\right)e^{-\left(\frac{ni\pi}{l}+1\right)x}\right]_0^l$$

$$= \frac{l^2.e^{(ni\pi-l)}}{i(ni\pi-l)^3}\left((ni\pi-l)^2 - 2(ni\pi-l)+2\right) + \frac{l^2.e^{-(ni\Pi+1)}}{i(ni\pi+l)^3}\left((ni\pi+l)^2 + 2(ni\pi+l)+2\right)$$

و: $\begin{cases} F(x) = 0 \\ g(x) = x\sin(x) \end{cases}$

$$: f(x) = 0 \rightarrow A_n = 0$$

$$: B_n = \frac{2}{nc\pi}\int_0^l x.\sin(x).\sin\left(\frac{n\pi}{l}x\right)dx = \frac{1}{nc\pi}\int_0^l \left[x\cos(1-\frac{n\pi}{l})x - x\cos(1+\frac{n\pi}{l})x\right]dx$$

$$= \frac{1}{nc\pi}\left[\frac{lx.\sin(1-\frac{n\pi}{l})x}{l-n\pi} + \frac{l^2.\cos(1-\frac{n\pi}{l})x}{(l-n\pi)^2} - \frac{lx.\sin(1+\frac{n\pi}{l})x}{l+n\pi} - \frac{l.\cos(1+\frac{n\pi}{l})x}{(l+n\pi)}\right]_0^l$$

$$= \frac{l^2}{nc\pi}\left[\frac{(l-n\pi)\sin(l-n\pi)+\cos(l-n\pi)-1}{(l-n\pi)^2} - \frac{(l+n\pi)\sin(l+n\pi)+\cos(l+n\pi)-1}{(l+n\pi)^2}\right] \quad ,l \neq n\pi$$

$: if \ : L = n\pi$

$$B_n = \frac{2}{nc\pi}\int_0^l \left[x.\sin^2(x)\right]dx = \frac{1}{nc\pi}\int_0^l \left[x - x\cos(2x)\right]dx$$

$$= \frac{1}{nc\pi}\left[\frac{1}{2}x^2 - \frac{1}{2}x\sin(2x) - \frac{1}{4}\cos(2x)\right] = \frac{2n^2\pi^2 - 4}{4nc\pi}$$

$$\Rightarrow B_n = \begin{cases} \frac{l}{nc\pi}\left[\frac{(l-n\pi)\sin(l-n\pi)+\cos(l-n\pi)-1}{(l-n\pi)^2} - \frac{(l+n\pi)\sin(l+n\pi)+\cos(l+n\pi)-1}{(l+n\pi)^2}\right] & : l \neq n\pi \\ \frac{2n^2\pi^2-4}{4nc\pi} & : l = n\pi \end{cases}$$

2-ب)

$$C^2 u_{xx} + \cosh\frac{x}{2} = u_{tt} \qquad 0 < x < 3, t > 0$$

$$\begin{cases} u(0,t) = 1 \\ u(3,t) = 4 \end{cases} \qquad \begin{cases} u(x,0) = x^2 + 1 \\ u_t(x,0) = 0 \end{cases}$$

$$C(x,t) = W(x,t) + v(x)$$

$$C^2\left[W_{xx} + \frac{d^2v}{dx^2}\right] + \cosh\frac{x}{2} = W_{tt} \to C^2\frac{d^2v}{dx^2} + \cosh\frac{x}{2} = 0 \to \frac{d^2v}{dx^2} = -\frac{e^{x/2} + e^{-x/2}}{2c}$$

شرط همگن بودن

$$V(x) = -\frac{2}{C^2}(e^{-x/2} + e^{-x/2}) + A_1 x + A_2 = -\frac{2}{c^2}(e^{-x/2} + e^{x/2}) + \left\{\frac{2}{3c^2}(e^{-3/2} + e^{3/2}) - \frac{4}{3c^2} + 1\right\}x + \frac{4}{c^2} + 1$$

$$u(0,t) = W(0,t) + V(0) \rightarrow V(0) = -\frac{4}{c^2} + A_2 = 1 \rightarrow A_2 = 1 + \frac{4}{c^2}$$

$$1 = V(0)$$

$$u(3,t) = W(3,t) + V(3) \rightarrow V(3) = -\frac{2}{c^2}(e^{-3/2} + e^{3/2}) + \frac{4}{c^2} + 1 + 3A_1 = 4$$

$$4 = V(3) \qquad A_1 = \frac{2}{3c^2}(e^{-3/2} + e^{3/2}) - \frac{4}{3c^2} + 1$$

$$C^2 W_{xx} = W_{tt}$$

$$\begin{cases} W(0,t) = 0 \\ W(3,t) = 0 \end{cases} \qquad \begin{cases} W(x,0) = x^2 + 1 - v(x) \\ W_t(x,0) = 0 \end{cases}$$

$$W(x,t) = M(x)N(t) \rightarrow C^2 M''N = N''M \rightarrow \frac{M''}{N} = \frac{N}{C^2 N} = \lambda = -\beta^2$$

$$M'' + \beta^2 M = 0 \rightarrow M(x) = A_1 \sin\beta x + A_2 \cos\beta x$$

$$N'' + \beta^2 C^2 N = 0 \rightarrow N(t) = B_1 \sin\beta ct + B_2 \cos\beta ct$$

$$M(0) = 0 \rightarrow A_2 = 0 \qquad M(3) = 0 \rightarrow \sin(\beta(3)) = \sin n\pi = 0 \rightarrow \beta_n = \frac{n\pi}{3}$$

$$w_n(x,t) = A_1 \sin\frac{n\pi}{3}x(B_1 \sin\frac{cn\pi}{3}t + B_2 \cos\frac{cn\pi}{3}t)$$

$$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{3}x(A_n \cos\frac{cn\pi}{3}t + B_n \sin\frac{cn\pi}{3}t)$$

$$w(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{3}x = x^2 + 1 - v(x) \rightarrow A_n = \frac{2}{3}\int_0^3 (x^2 + 1 - v(x)) \sin\frac{n\pi}{3xdx}$$

$$\frac{\delta w}{\delta t} = w = \sum_{n=1}^{\infty} \frac{cn\pi}{3} \sin\frac{n\pi}{3}x\left(-A_n \sin\frac{cn\pi}{3}t + B_n \cos\frac{cn\pi}{3}t\right)$$

$$w_t(x,0) = 0 = \sum_{n=1}^{\infty} B_n . \frac{en\pi}{3} \sin\frac{n\pi}{3}x \rightarrow B_n = 0$$

$$w(x,t) = \sum_{n=1}^{\infty} \left\{\frac{2}{3}\int_0^3 (x + 1 - v(x)) \sin\frac{n\pi}{3} xdx\right\} \sin\frac{n\pi}{3}x \cos\frac{cn\pi}{3}t$$

$c^2 u_{xx} + \cos^2 x = u_{tt} \qquad 0 < x < 3, t > 0$

$\begin{cases} u(0,t) = 2 \\ u(3,t) = 14 \end{cases} \qquad \begin{cases} u(x,0) = x^2/4 \\ u_t(x,t) = 0 \end{cases}$

$u(x,t) = w(x,t) + v(x)$

$c^2\left[w_{xx} + \frac{d^2 v}{dx^2}\right] + \cos x = w_{tt}$ شرط همگن

بودن $\rightarrow c^2 \frac{d^2 v}{dx^2} + \cos^2 x = 0 \rightarrow \frac{d^2 v}{dx^2} = -\frac{1}{c^2}\cos^2 x$

$\rightarrow \frac{d^2 v}{dx^2} = -\frac{1}{2c^2}(1 + \cos 2x) \rightarrow \frac{dv}{dx} = -\frac{1}{2c^2}(x + 1/2 \sin 2x) + c_1$

$\rightarrow v(x) = -\frac{1}{2c^2}\left(\frac{x^2}{2} - 1/4 \cos 2x\right) + c_1 x + c_2$

$\begin{cases} v(0) = 2 \\ v(3) = 14 \end{cases} \qquad \begin{aligned} &2 = -\frac{1}{2c^2}\left(1/4\right) + c_2 \rightarrow c_2 = 2 - \frac{1}{8c^2} \\ &14 = -\frac{1}{2c^2}\left(\frac{9}{2} - 1/4 \cos 6\right) + 3c_1 + 2 - \frac{1}{8c^2} \rightarrow c_1 = 4 + \frac{19 - \cos 6}{24c^2} \end{aligned}$

$v(x) = -\frac{1}{2c^2}\left(\frac{x^2}{2} - 1/4 \cos 2x\right) + \left(4 + \frac{19 - \cos 6}{24c^2}\right)x + 2 - \frac{1}{8c^2}$

$c^2 w_{xx} = w_{tt}$

$\begin{cases} w(0,t) = u(0,t) - v(0) = 0 \\ w(3,t) = u(3,t) - v(3) = 0 \end{cases} \qquad \begin{cases} w(x,0) = u(x,0) - v(x) = \frac{x^2}{4} - v(x) \\ w_t(x,0) = u_t(x,0) = 0 \end{cases}$

$w(x,t) = M(x)N(t)$

$c^2 M'' N = MN'' \rightarrow \frac{M''}{M} = \frac{N''}{C^2 N} = -\beta^2 \begin{cases} M'' + \beta^2 M = 0 \\ N + \beta^2 C^2 N = 0 \end{cases}$

$M(0) = A_1 \sin \beta x + A_2 \cos \beta x$

$M(0) = 0 \rightarrow A_2 = 0 \qquad M(3) = 0 \rightarrow \sin 3\beta = 0 = \sin n\Pi \rightarrow \beta_n = \frac{n\Pi}{3}$

$$N(T) = D_1 \sin \beta ct + D_2 \cos \beta ct$$

$$W_n(x,t) = A_1 \sin \frac{n\pi}{3} x (D_2 \cos \beta ct + D_1 \sin \beta ct)$$

$$W(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{3} x \left(A_n \cos \frac{cn\pi}{3} t + B_n \sin \frac{cn\pi}{3} t \right)$$

$$W(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{3} x \to A_n = \frac{2}{3} \int_0^3 \left\{ x^2 / 4 - v(x) \right\} \sin \frac{n\pi}{3} x dx$$

$$\frac{\delta W}{\delta t} = W_t = \sum_{n=1}^{\infty} \frac{cn\pi}{3} \sin \frac{n\pi}{3} x \left(-A_n \sin \frac{cn\pi}{3} t + B_n \cos \frac{cn\pi}{3} t \right)$$

$$W_t(x,0) = 0 = \sum_{n=1}^{\infty} B_n . \frac{cn\pi}{3} \sin \frac{n\pi}{3} x \to B_n = 0$$

$$W(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{2}{3} \int_0^3 \left(x^2 / 4 - v(x) \right) \sin \frac{n\pi}{3} x dx \right\} \sin \frac{n\pi}{3} x \cos \frac{cn\pi}{3} t$$

$$u(x,t) = W(x,t) + V(x)$$

پاسخ سوال 3 قسمت الف:

$$c^2 \nabla^2 u = u_{tt} \qquad 0 < x < a, 0 < y < b, t > 0$$

$$\begin{cases} u = 0 & \text{روی تمام مرزها} \\ u(x,y,0) = y^3 \\ u_t(x,y,0) = e^{e+y} \end{cases}$$

$$u(x,y,t) = M(x).N(y).P(t)$$

جایگذاری در معادله: $\rightarrow c^2 (M'' NP + MN'' P) = MNP''$

$$\frac{M''}{M} = \frac{P''}{C^2 P} = \lambda < 0$$

$$P'' - C^2 \lambda P = 0 \Rightarrow \underline{P(t) = A_1 \sin c\sqrt{\lambda} t + A_2 \cos c\sqrt{\lambda} t}$$

$$\frac{M''}{M} + \frac{N''}{N} = \lambda \Rightarrow \frac{M''}{M} = \lambda - \frac{N''}{N} = \mu < 0$$

$$M'' - \mu M = 0 \Rightarrow M(x) = B_1 \sin \sqrt{\mu} x + B_2 \cos \sqrt{\mu} x$$

$$M(0)=0 \Rightarrow B_2=0 \rightarrow B_1 \neq 0$$

$$M(a)=0 \Rightarrow B_1 \sin\sqrt{\mu}a=0 \Rightarrow \sqrt{\mu}a=n\pi \Rightarrow \sqrt{\mu_n}=\frac{n\pi}{a}, n=1,2,3,\ldots$$

پس: $M(x)=B_1 \sin\frac{n\pi}{a}x$

$$\lambda-\frac{N''}{N}=\mu \rightarrow \frac{N''}{N}=\lambda-\mu=\beta<0$$

$$N''-\beta N=0 \rightarrow N(y)=C_1 \sin\sqrt{\beta}y+C_2 \cos\sqrt{\beta}y$$

$$N(0)=0 \Rightarrow C_2-0 \rightarrow C_1 \neq 0$$

$$N(b)=0 \Rightarrow C_1 \sin\sqrt{B}b=0 \Rightarrow \sqrt{B_m}b=m\pi \Rightarrow \sqrt{B_m}=\frac{m\pi}{b}, m=1,2,3,\ldots$$

پس: $N(y)=C_1 \sin\frac{m\pi}{b}y$

$$\lambda-\mu=\beta \Rightarrow \lambda=\beta+\mu \Rightarrow \sqrt{\lambda}=\sqrt{\beta+\mu}=\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}$$

پس: $U(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi}{a}x.\sin\frac{m\pi}{b}y\left(A_{nm}\sin c\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}t+B_{nm}\cos c\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}t\right)$

$$U(x,y,0)=y^3\cos 3x=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{nm}\sin\frac{n\pi}{a}x.\sin\frac{m\pi}{b}y$$

$$\Rightarrow B_{nm}=\frac{4}{ab}\int_0^b\int_0^a y^3\cos 3x.\sin\frac{n\pi}{a}x.\sin\frac{m\pi}{b}y\,dxdy$$

$$\Rightarrow B_{nm}=\frac{4}{ab}\left(\int_0^b y^3\sin\frac{m\pi}{b}y\,dy\right)\left(\int_0^a \cos 3x\sin\frac{n\pi}{a}x\,dx\right)$$

$$\Rightarrow B_{nm}=\frac{4}{ab}\left(\frac{-b}{m\pi}y^3\cos\frac{m\pi}{b}y+\frac{3b^2}{m^2\pi^2}y\sin\frac{m\pi}{b}y+\frac{6b^3}{m^3\pi^3}y\cos\frac{m\pi}{b}y-\frac{6b^4}{m^4\pi^4}\sin\frac{m\pi}{b}y\right)_0^b$$

$$\times\frac{1}{2}\int_0^a\left(\sin.\left(\frac{n\pi}{a}-3\right)x+\sin\left(\frac{n\pi}{a}+3\right)x\right)dx$$

$$\Rightarrow B_{nm}=2b^3(-1)^m\left(\frac{6}{m^3\pi^3}-\frac{1}{m\pi}\right)\left(\frac{-\cos(n\pi-3a)+1}{n\pi-3a}-\frac{\cos(n\pi+3a)-1}{n\pi+3a}\right)$$

$$U(x.y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin.\frac{n\pi}{a}x.\sin\frac{m\pi}{b}y\left(A_{nm}C\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\cos C\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}t\right.$$

$$\left.-BC\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\sin C\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}t\right)$$

$$U(x,y,0)=e^{x+y}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}C\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\sin\frac{n\pi}{a}x.\sin\frac{m\pi}{b}y$$

$$C_{nm}$$

$$C_{nm}=\frac{4}{ab}\int_0^b\int_0^a e^x\times e^y\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}ydxdy=\frac{4}{ab}\int_0^b e^y\sin\frac{m\pi}{b}ydy.\int_0^a e^x\sin\frac{n\pi}{a}xdx$$

$$\Rightarrow C_{nm}=\frac{4}{ab}\times\frac{bm\pi\left(1-e^b(-1)^m\right)}{m^2\pi^2+b^2}\times\frac{an\pi\left(1-e^a(-1)^n\right)}{n^2\pi^2+a^2}$$

$$\Rightarrow A_{nm}=\frac{4\pi^2nm}{C\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}}\times\frac{\left(1-e^6(-1)^m\right)\left(1-e^a(-1)^n\right)}{\left(m^2\pi^2+b^2\right)\left(n^2\pi^2+a^2\right)}$$

پاسخ 3 قسمت ب:

$$U(x,y,t)=M(x).N(y).P(t)$$

$$\frac{M\backslash'}{M}+\frac{N''}{N}=\frac{P''}{C^2P}=-\lambda^2$$

$$-\frac{M''}{M}=-\beta^2\to M''+\beta^2M=0\to M(x)=A/\sin\beta x+A\cos\beta x$$

$$M'(x)=\beta\left(A_1\cos\beta x-A_2\sin\beta x\right)$$

$$M'(0)=0\to\beta A_1=0\Rightarrow\begin{cases}\beta=0\\A_1=0\to A_2\neq0\end{cases}$$

$$M'(a)=0\to-A_2\beta\sin\beta a=0\to\begin{cases}\beta=0\\\sin\beta A=0=\sin M\pi\to\beta_M=\frac{M\pi}{A}\quad M=1,2,3,\ldots\end{cases}$$