با توجه به اینکه u در جهت y متناوب است پس

ضریب Nباید مثبت باشد یعنی $\lambda > 0$

$$\lambda = \beta^2$$

$$N(y) = A_1 \sin \beta y + A_2 \cos \beta y$$

$$u(x,0) = 0 \rightarrow n(0) = 0 \rightarrow A_2 = 0 \qquad \rightarrow N(y) = A_1 \sin \frac{n\pi}{a} y$$

$$u(x,0) = 0 \rightarrow N(a) = 0 \rightarrow \begin{cases} A_1 \neq 0 \\ \sin \beta a = 0 \rightarrow \sin \beta a = \sin n\pi \rightarrow \beta_n = \frac{n\pi}{a}, n = 1,2,\ldots \end{cases}$$

$$M(x) = A'_1 e^{\beta x} + A'_2 e^{-\beta x}$$

از فیزیک مساله درمی یابیم که وقتی: $\lim u(x,y) < \infty$

پس ضریب $A'_1 = 0$

$$x \rightarrow \infty$$

$$M(x) = A'_2 e^{-\beta x}$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} y . e^{-\frac{n\pi}{a} x}$$

سری فوریه سینوسی $e^{-y} \sin y$

$$u(0,y) = e^{-y} \sin y = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} y$$

$$A_n = \frac{2}{a} \int_0^a \left(e^{-y} \sin y\right) \sin \frac{n\pi}{a} y dy = \frac{1}{a} \int_0^a e^{-y} \left[\cos\left(\frac{n\pi}{a} + 1\right) y - \cos\left(\frac{n\pi}{a} - 1\right) y \right] dy$$

$$= \frac{1}{a}\left[\frac{1}{\frac{n\pi}{a}+1} e^{-y}\sin\left(\frac{n\pi}{a}+1\right) - \frac{1}{\left(\frac{n\pi}{a}+1\right)^2} e^{-y}\cos\left(\frac{n\pi}{a}+1\right) y \right.$$

$$\left. - \frac{1}{\frac{n\pi}{a}-1} e^{-y}\sin\left(\frac{n\pi}{a}-1\right)y + \frac{e^{-y}}{\left(\frac{n\pi}{a}-1\right)^2}\cos\left(\frac{n\pi}{a}-1\right) \right|_0^a$$

$$+ \frac{1}{a\left(\frac{n\pi}{a}-1\right)^2}\int_0^a e^{-y}\cos\left(\frac{n\pi}{a}-1\right)y\,dy - \frac{1}{a\left(\frac{n\pi}{a}+1\right)^2}\int_0^a e^{-y}\cos\left(\frac{n\pi}{a}+1\right)y\,dy$$

	D	I
+	e^{-y}	$Cosmy$
−	$-e^{-y}$	$\frac{1}{m}Sinmy$
+	e^{-y}	$-\frac{1}{m^2}Cosmy$
		$-\frac{1}{m^2}\int e^{-y}Cosmy$

$$A_n=\frac{\left(\frac{n\pi}{a}+1\right)}{a\left[\left(\frac{n\pi}{a}+1\right)^2+1\right]}\left[e^{-a}\sin\left(\frac{n\pi}{a}+1\right)a-\frac{1}{\frac{n\pi}{a}+1}e^{-a}\cos\left(\frac{n\pi}{a}+1\right)a+\frac{1}{\frac{n\pi}{a}+1}\right]$$

$$-\frac{\left(\frac{n\pi}{a}-1\right)}{a\left[\left(\frac{n\pi}{a}-1\right)^2-1\right]}\left[e^{-a}\sin\left(\frac{n\pi}{a}-1\right)a-\frac{1}{\frac{n\pi}{a}-1}e^{-a}\cos\left(\frac{n\pi}{a}-1\right)a+\frac{1}{\frac{n\pi}{a}-1}\right]$$

$$=\frac{\left(\frac{n\pi}{a}+1\right)}{a\left[\left(\frac{n\pi}{a}+1\right)^2+1\right]}\left[(-1)^n e^{-a}\sin a+\frac{(-1)^{n+1}}{\frac{n\pi}{a}+1}e\cos a+\frac{1}{\frac{n\pi}{a}+1}\right]-\frac{\left(\frac{n\pi}{a}-1\right)}{a\left[\left(\frac{n\pi}{a}-1\right)^2-1\right]}$$

$$\left[(-1)^{n+1}e^{-a}\sin a+\frac{(-1)^{n+1}\cos a}{\frac{n\pi}{a}-1}e^{-a}+\frac{1}{\frac{n\pi}{a}-1}\right]$$

ب) $u(x,0)=0$ $\qquad u(x,a)=u_\circ$ $\qquad u(0,y)=0$

u در جهت y غیر همگن می باشد.

$$u(x,y)=W(x,y)+r(y)$$

$$u_{xx}=W_{xx} \qquad ,u_{yy}=w_{yy}+r_{rr}$$

$$w_{xx}+w_{yy}+r_{yy}=0\rightarrow r_{yy}=0$$

$$\rightarrow r_y=A\rightarrow r(y)=Ay+B$$

$$\begin{cases} r(0)=0\rightarrow B=0 \\ r(a)=u_\circ\rightarrow A=\frac{u_\circ}{a}\end{cases}\rightarrow r(y)=\frac{u_\circ}{a}y$$

معادله خطی و همگن

$$W(x,0)=u(x,0)-r(0)=0$$

$$W(x,0)=u(x,,0)-r(a)=u_\circ-\frac{u_\circ}{a}\times a=0$$

و شرایط موزی در جهت yهمگن است

$$w(0,y)=u(0,y)-r(y)=-\frac{u_\circ}{a}y$$

w در جهت y متناوب است

$$W(x,y)=M(x)N(y)$$

پس ضریب N مثبت است

$$\frac{M''}{M}+\frac{N''}{N}=0\rightarrow\frac{M''}{M}=-\frac{N''}{N}=\lambda\begin{cases}M''-\lambda M=0\\N''+\lambda N=0\end{cases}$$

یعنی $\lambda>0\ (\lambda=\beta^2)$

$$N=A_1\sin\beta y+A_2\cos\beta y$$

$$N(0)=0\rightarrow A_2=0 \qquad ,N(a)=0\rightarrow \begin{matrix}A_1\neq 0\\ \sin\beta a=0\rightarrow\beta_n=\frac{n\pi}{a},n=1,2,\ldots\end{matrix}$$

$$M(x)=A'_1e^{-\beta x}+A'_2e^{\beta x}$$

با توجه به شرایط کراندار u وقتی $x\rightarrow\infty$، ضریب

$$A'_2=0$$

$$M(x)=A_1e^{-\frac{n\pi}{a}x}$$

$$w(x,y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{a}y.e^{-\frac{n\pi}{a}x}$$

سری فوریه سینوسی تابع $-\frac{u_\circ}{a}y$

$$W(0,y)=-\frac{u_\circ}{a}y=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{a}y$$

	D	I
+	y	$Sin\frac{n\pi}{a}y$
−	1	$-\frac{a}{n\pi}Cos\frac{n\pi}{a}y$
	0	$-\left(\frac{a}{n\pi}\right)^2Sin\frac{n\pi}{a}y$

$$A_n = \frac{2}{a}\int_0^a \left(-\frac{u_\circ}{a}y\right)\sin\frac{n\pi}{a}y\,dy = -\frac{2u_\circ}{a^2}\left[-\frac{a}{n\pi}y\cos\frac{n\pi}{a}y + \left(\frac{a}{n\pi}\right)^2\sin\frac{n\pi}{a}y\right]_0^a = \frac{2u_\circ}{an\pi}a\cos n\pi = \frac{2u}{n\pi}(-1)^n$$

$$u(x,y) = e(x,y) + r(y) = \frac{u_\circ}{a}y + \frac{2u_\circ}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{a}y.e^{-\frac{n\pi}{a}x}$$

ج) $u_y(x,0)=0 \quad , u(x,a)=0 \qquad u(0,y)=y^2$

شرایط مرزی در جهت y همگن است.

$$u(x,y)=M(x)N(y)$$

حالات مختلف λ را بررسی می کنیم:

$$\frac{M''}{M} = -\frac{N''}{N} = \lambda$$

$\lambda<0 : \lambda=-\beta^2$

$N = A_1 e^{\beta y} + A_2 e^{-\beta y} \to N'(y) = A_1\beta e^{\beta y} - A_2\beta e^{-\beta y}$

$$N'(0)=0 \to \begin{cases} \beta=0 \to \\ A_1=A_2 \end{cases}$$

جداگانه بررسی میشود

جواب بدیهی

$$N(a)=0 \to A_1(e^{\beta a}-e^{-\beta a})=0 \to -2A_1\sinh\beta^{\neq 0}_{a=0} \to A_1=0$$

($\sinh y$ فقط به ازای $y=0$ صفر است)

$\lambda=0 \to N=AY+B \to N_y=A \to N_y=0 \to A=0$

$N(a)=0 \to B=0$ جواب بدیهی

$\lambda>0 : \beta^2=\lambda$

$N = A_1\sin\beta y + A_2\cos\beta y \to N_y = A_1\beta\cos\beta y - A_2\beta\sin\beta y$

$$N_y(0)=0 \to \begin{cases} A_1=0 \\ \beta\neq 0 \end{cases}$$

$$N(a)=0 \to A_2\cos\beta a=0 \xrightarrow{A_2\neq 0} \cos\beta a = \cos\left(\frac{2n-1}{2}\right)\pi \to \beta=\frac{2n-1}{2a}\pi, n=1,2,...$$

$$N(y) = A_2 \cos\left(\frac{2n-1}{2a}\right)\pi y$$

$$M(x) = A'_1 e^{-\beta x} + A'_2 e^{\beta x}$$

$$* \lim u(x,y) < \infty \rightarrow A_2' = 0$$

$$x \rightarrow \infty$$

$$u = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{2n-1}{2a}\pi\right)x} \cos\left(\frac{2n-1}{2a}\right)\pi y$$

$$u(0,y) = y = \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n-1}{2a}\right)\pi y$$ y^2 سری فوریه

کسینوسی

$$A_n = \frac{2}{a}\int_0^a y^2 \cos\left(\frac{2n-1}{2a}\right)\pi y dy = \frac{2}{a}\left[\frac{2a}{(2n-1)\pi} y^2 \sin\left(\frac{2n-1}{2a}\right)\pi y + \left(\frac{2a}{(2n-1)\pi}\right)^2 2y\cos\left(\frac{2n-1}{2a}\right)\pi\right.$$

$$\left. -2\left(\frac{2a}{(2n01)\pi}\right)^3 \sin\left(\frac{2n-1}{2a}\right)\pi y\right|_0^a = \frac{2}{a}\left[\frac{2a^3}{(2n-1)\pi}(-1)^{n+1} - 2\left(\frac{2a}{(2n-1)\pi}\right)^3 (-1)^{n+1}\right]$$

د) $u_y(x,0) = u_y(x,a) = 0$ $u(0,y) = \infty shy$

شرایط مرزی در جهت yهمگن است. با استفاده از روش جداسازی متغیرها داریم:

$$u(x,y) = M(x)N(y)$$

مشتق u در جهت y متناوب است پس Nمتناوب است

و $\lambda > 0$ $\frac{M''}{M} = -\frac{N''}{N} = \lambda$

$$\lambda = \beta^2 \rightarrow N = A_1 \cos\beta y + A_2 \sin\beta y$$

بررسی

$$N_y = -A_1\beta \sin\beta y + A_2\beta\infty\beta y \begin{cases} N_y(0) = 0 \rightarrow \begin{cases}\beta = 0 \\ A_2 = 0\end{cases} \\ n_y(a) = 0 \rightarrow A_1 \neq 0, \beta_n = \frac{n\pi}{a} n = 1,2,\ldots \end{cases}$$

خواهد شد

$$*\lim u(x,y)<\infty$$

$$M(x)=A'_1 e^{-\beta x} A'_2 e^{\beta x}$$

$$x\to\infty$$

$$\beta=0\to N=Ay+B\to N_y=A\begin{cases} ny(0)=0 \\ Ny(a)0=0 \end{cases}\to A=0$$

$$M=A'x+B'$$

* بدلیل شرط $\lim u(x,y)<\infty$ ضریب $A'=0$

$$x\to\infty$$

$$u(x,y)=A+\sum_{n=1}^{\infty}A_n e^{-\frac{n\pi}{a}x}.\cos\frac{n\pi}{a}y$$

$$u(0,y)=\cosh y=A+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi}{a}y$$ سری فوریه $\cosh y$

$$A=\frac{1}{a}\int_0^a\cosh y dy=\frac{1}{a}\sinh y\Big|_0^a=\frac{1}{a}\sinh a$$

کسینوسی

$$A_n=\frac{2}{a}\int_0^a\cosh.\cos\frac{n\pi}{a}ydy=\frac{1}{a}\int_0^a\left[\cos\left(\frac{n\pi}{a}+i\right)y+\cos\left(\frac{n\pi}{a}-i\right)y\right]dy$$

$$=\frac{1}{a}\left[\frac{1}{\frac{n\pi}{a}+1}\sin\left(\frac{n\pi}{a}+i\right)y+\frac{1}{\frac{n\pi}{a}-i}\sin\left(\frac{n\pi}{a}-i\right)\right]\Bigg|_0^a=\frac{1}{a}\left[\frac{(-1)}{\frac{n\pi}{a}+i}\sin\left(\frac{n\pi}{a}+i\right)a\right.$$

$$\left.+\frac{1}{\frac{n\pi}{a}-i}\sin\left(\frac{n\pi}{a}-i\right)a\right]=\frac{1}{a}\left[\frac{(-1)^n}{\frac{n\pi}{a}+i}i\sinh a-\frac{(-1)^n}{\frac{n\pi}{a}-i}i\sinh a\right]$$

$$*\cosh y=\cos(iy)$$
$$\sin iy=i\sinh y$$
$$,\sin(n\pi+ia)=(-1)^n\sin ia$$

و) $u(x,0)=0, u_y(x,a)=0$ $\qquad u(0,y)=4y$

شرایط مرزی در جهت yهمگن است:

$$u''(x,y)=M(x)N(y)$$

$$\frac{M''}{M} = -\frac{N''}{N} = \lambda$$

مقادیر مختلف λ را بررسی می کنیم:

$\lambda < 0 : \lambda = -\beta^2$

$N = A_1 e + A_2 e^{\beta Y} \rightarrow N_Y = -A_1 \beta e^{-\beta Y} + A_2 \beta e^{\beta Y}$

$N(0) = 0 \rightarrow A_1 = -A_2$

$N_y(a) = 0 \rightarrow -\beta A_1 (2\cosh \beta^{\neq 0} a) = 0 \rightarrow \begin{cases} A_1 = 0 \rightarrow A_2 = 0 \\ \beta \neq 0 \end{cases}$ جواب بدیهی

$\lambda = 0 \rightarrow N = Ay + B \rightarrow N_y = A$

$N(0) = 0 \rightarrow B = 0$ جواب بدیهی

$N_y(a) = 0 \rightarrow A = 0$

$\lambda > 0 : (\lambda = \beta^2)$

$N = A_1 \sin \beta y + A_2 \cos \beta y \rightarrow N_y = A_1 \beta \cos \beta y - A_2 \beta \sin \beta y$

$N(0) = 0 \rightarrow A_2 = 0$

$N(a) = 0 \rightarrow A_1 \beta \neq 0 \rightarrow \cos \beta a = 0 \rightarrow \cos \beta a = \cos \frac{2n-1}{2} \pi \rightarrow \beta_n = \frac{2n-1}{2a} \pi, n = 1,2,...$

*شرایط کرانداری u در $\infty \rightarrow A'_2 = 0$

$M(x) = A'_1 e^{-\beta x} + A'_2 e^{\beta x}$

$$u(x,y) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{2n-1}{2a}\pi\right) a} \sin\left(\frac{2n-1}{2a}\right) \pi y$$

سوال 13 قسمت (و)

$\nabla^2 = 0 \qquad 0 < y < a \qquad , x > 0$

$u(x,0) = 0 \qquad u_y(x,0) = 0 \qquad u(0,y) = 4y \qquad \lim_{x \rightarrow \infty} u = 0$

$$u(w,y) = \int_0^{\infty} u(x,y) \sin wx dx$$

$$F_s\{u_{xx}(x,y)\} = \int_0^{\infty} u_{xx}(x,y) \sin wx dx \begin{cases} u_{xx} dx = dV, V = u_x \\ \sin wx = Z, dz = w \cos wx \end{cases}$$

$$= u_x \sin wx \,\Big]_0^{\infty} - w\int_0^{\infty} u_x \cos wx dx \begin{cases} u_x dx = dv & V = u \\ z = \cos wx & dz = -w \sin wx \end{cases}$$

$$= -w\left[u \cos wx \Big|_0^{\infty} + w\int_0^{\infty} u \sin wx dx \right]$$

$$= -w^2 u(w,y) + 4wy$$

$$F_s(u_{yy}(x,y)) = u_{yy}(w,y) \to 4wy - w^2 u(w,y) + \frac{5^2}{5y^2} u(w,y) = 0$$

$D^2 - w^2 = 0$ معادله مشخصه $\to D = \pm w$

$u_h = c_1 e^{wy} + c_2 e^{-wy}$ $\qquad u_p = Ay + B$

$$\to 0 - Aw^2 y - Bw^2 = -4wy \to B = 0$$

$$\to Aw^2 = 4w \to A = \frac{4}{w}$$

$$\to u(w,y) = c_1 e^{wy} + c_2 e^{-wy} + \frac{4}{w} y$$

$$\to u(w,0) = c_1 + c_2 = \int_0^{\infty} u(x,0) \sin wx dx = 0 \to c_1 = -c_2$$

$$\to u_y(w,y) = \int_0^{\infty} u_y(x,y) \sin wx dx = c_1 w e^{wy} + c_1 w e^{-wy} + \frac{4}{w}$$

$$u_y(w,a) = c_1 w(e^{wa} + e^{-wa}) + \frac{4}{w} = \int_0^{\infty} u(x,a) \sin wx dx = 0$$

$$\to c_1 = \frac{-4}{w^2(e^{wa} + e^{-wa})} \qquad , C_2 = \frac{4}{w^2\left(e^{wa} + e^{-wa}\right)}$$

$$u(w,y) = \frac{(-4)}{w^2(e^{wa} + e^{-wa})} e^{wy} + \frac{4}{w^2(e^{wa} + e^{-wa})} e^{-wy} + \frac{4}{w} y$$

$$u(x,y) = \frac{2}{\pi} \int_0^{\infty} \left\{ \frac{(-4)(2)}{w(e^{wa} + e^{-wa})} \sinh(wg) + \frac{4}{w} y \right\} \sin wx dw$$

$4y$ سری فوریه سینوسی

$$u(0,y) = 4y = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2a}\right) \pi y$$

$$A_n=\frac{2}{a}\int_0^a 4y\sin\left(\frac{2n-1}{2a}\right)\pi y dy=\frac{8}{a}\left[-\frac{2a}{(2n-1)}y\cos\left(\frac{2n-1}{2a}\right)\pi y+\left(\frac{2a}{(2n-1)\pi}\right)^2\sin\left(\frac{2n-1}{2a}\right)\pi y\right]\Bigg|_0^a$$

$$=\frac{32a}{((2n-1)\pi)^2}\sin\left(\frac{2n-1}{2a}\right)\pi a=\frac{32a}{((2n-1)\pi)^2}(-1)^{n+1}$$

ز) $u(x,0)=u_1$ $\qquad$ $u(x,0)=u_2$ $\qquad$ $u(0,y)=f(y)$

جواب قسمت همگن

شرایط موزی غیر همگن می باشد:

$$u(x,y)=w(x,y)+r(y)$$

جواب خصوصی

$$u_{xx}=W_{xx}$$
$$u_{yy}=W_{yy}+r_{yy}$$
$$W_{xx}+W_{yy}+r_{yy}=0$$
$$r_{yy}=0\rightarrow r_y=A\rightarrow r=Ay+B$$

$$r(0)=u_1\rightarrow B=u_1$$
$$r(a)=u_2\rightarrow A=\frac{u_2-u_1}{a} \qquad \rightarrow r(y)=\frac{u_2-u_1}{a}y+B_1$$

$$w_{xx}+w_{yy}=0$$
$$w(x,0)=u(x,0)-r(0)=u_1-u_1=0$$
$$w(x,a)=u(x,a)-r(a)=0$$
$$w(0,y)=u(0,y)-r(y)=f(y)-r(y)$$

معادله خطی و همگن و شرایط موزی همگن: جداسازی متغیرها

$$W(x,y)=MZ(x)N(y)$$

w در جهت y متناوب ← پس N متناوب است ← $\lambda>0$

$$\frac{M''}{M}+\frac{N''}{N}=0\rightarrow\frac{M''}{M}=-\frac{N''}{N}=\lambda$$

$$\lambda=\beta^2$$
$$N(y)=A_1\sin\beta y+A_2\cos\beta y$$