$N(0)=0 \rightarrow A_2=0, N(a)=0 \rightarrow A_1 \sin\beta a=0 \xrightarrow{A_1\neq 0} \beta_n=\frac{n\pi}{a}, n=1,2,..$

$M(x)=A'_1 e^{-\beta x}+A'_2 e^{\beta x}$ $A'_2=0 \leftarrow x\rightarrow\infty$ شرط w در

کرانداری

$$w(x,y)=\sum_{n=1}^{\infty} A_n e^{-\frac{n\pi}{a}} \sin\frac{n\pi}{a}y$$

سری فوریه سینوسی

$$w(0,y)=f(y)-u_1-\frac{u_2-u_1}{a}y=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}y$$

$$A_n=\frac{2}{a}\int_0^a \left[f(y)-(u_1+\frac{u_2-u_1}{a}y)\right]\sin\frac{n\pi}{a}ydy$$

$$w(x,y)+r(y)=u(x,y)$$

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14)

$\nabla^2 u=0 \Rightarrow u_{xx}+u_{yy}=0$

$-\infty<x<+\infty, y>0$

الف) $u(x,0)=\begin{cases}1 & -1<x<1 \\ 0 & {\scriptstyle x>1,\ x<-1}\end{cases}$

$$F(u(x,y))=u(w,y)=\int_{-\infty}^{+\infty} u(x,y)e^{-iww}\ln$$

$u_{xx}+u_{yy}=0 \Rightarrow$ x تبدیل فوریه نسبت

به $\int_{-\infty}^{+\infty} u_{xx}(x,y)e^{-iwx}dx+\int_{-\infty}^{+\infty} u_{yy}(x,y)e^{-iwx}dx=0$

$\Rightarrow (iw)^2 u(w,y)+u_{yy}(w,y)=0 \Rightarrow -w^2 u(w,y)+u_{yy}(w,y)=0$

معادله دیفرانسیل را حل می کنیم:

معادله ی مشخصه $-w^2+t^2=0 \Rightarrow t=\pm w \Rightarrow u(w,y)=A_1e^{wy}+A_2e^{-wy}$

$\lim|u(x,y)|xm \Rightarrow u(w,y)=ce^{-|w|y}\begin{cases}c=A_2 & w>0\\ c=A_1 & w>0\end{cases}$

$x \to \pm \infty$

$y \to \infty$

$$\Rightarrow u(w,0)=c=F(w)=\int_{-\infty}^{+\infty}F(x)e^{-iwx}dx=\int_{-1}^{1}e^{-iwx}dx+\int_{-\infty}^{-1}(0)e^{-iwx}dx+\int_{1}^{\infty}(0)e^{-iwx}dx=\frac{1}{iw}e^{-iwx}\Big]_{-1}^{1}$$

$$=\frac{i}{w}(e^{-iw}-e^{iw})=-\frac{1}{w}(e^{iw}-e^{-iw})=-\frac{i}{w}\left(\frac{Sinw}{2i}\right)$$

$$c=-\frac{Sinw}{2w}\Rightarrow u(w,y)=\frac{-Sinw}{2w}e^{-|w|y}$$

تبدیل وارونه فوریه

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}u(w,y)e^{iwx}dw=\frac{1}{2\pi}\int_{-\infty}^{+\infty}-\frac{Sinw}{2w}e^{|w|y}e^{iwx}dw$$

$$u(x,y)=-\frac{1}{4\pi}\left[\int_{-\infty}^{0}\frac{\sin w}{w}e^{iwx}e^{wy}dw+\int_{0}^{\infty}\frac{\sin w}{w}e^{iwx}e^{-wy}dw\right]$$

ب) $u(x,0)=\begin{cases}K & -x<x<0\\ 0 & 0<x<a\\ e^{-|x|} & |x|>a\end{cases}$

$$F(u(x,y))=u(w,y)=\int_{-\infty}^{+\infty}u(x,y)e^{-iwx}dx$$

معادله دیفرانسیل مرتبه ی دوم

$$F(u_{xx}+u_{yy})=0\Rightarrow(iw)^2u(w,y)+u_{yy}(w,y)=0$$

معادله مشخصه $t^2-w^2=0\Rightarrow t=\pm w\Rightarrow u(w,y)=A_1e^{wy}+A_2e^{-wy}$

→

$$\lim|u(x,y)|<m\Rightarrow u(w,y)=ce^{-|w|y}=\begin{cases}c=A_2 & w>0\\ c=A_1 & w<0\end{cases}$$

$x \to \pm \infty$

$y \to \infty$

$$u(w,0) = c = F(w) = \int_{-\infty}^{+\infty} F(x) e^{-iwx} dx = \int_{-\infty}^{-\infty} e^{x} e^{-iwx} dx + \int_{-\infty}^{0} K e^{-iwx} dx$$

$$+ \int_{0}^{a} 0 e^{-iwx} dx + \int_{a}^{+\infty} e^{-x} e^{-iwx} dx$$

$$c = \int_{-\infty}^{-a} e^{(1-iw)} dx + K \int_{-a}^{0} e^{-iwx} dx + \int_{a}^{\infty} e^{(-iw-1)} dx$$

$$c = \frac{1}{1-iw} e^{x(1-iw)} \Big]_{-\infty}^{-a} + \frac{K}{-iw} e^{-iwx} \Big]_{-a}^{0} + \frac{1}{-iw-1} e^{(-iw-1)} \Big]_{a}^{\infty}$$

$$c = \frac{1}{1-iw}\left[e^{-a(-iw)} - 0\right] + \frac{iK}{w}(1 - e^{iwx}) - \frac{1}{1+iw}\left[0 - e^{-a(iw+1)}\right]$$

$$u(w,0) = c \Rightarrow u(w,y) = ce^{-|w|y}$$

تبدیل وارون $u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u(w,y) e^{iwx} dw = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ce^{|w|y} e^{iwx} dw$

$$\Rightarrow u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ce^{-|w|y} e^{iwx} dx$$

با جایگذاری c از رابطه ی بالا و انتگرال گیری $u(x,y)$ بدست می آید.

ج) $u(x,0) = e^{-x2}$ $\quad u(x,y) = ?$

$$F(u(x,y)) = F(w) = u(w,y) = -\int_{-\infty}^{+\infty} u(x,y) e^{-iwx} dx$$

تبدیل فوریه نسبت به x $\quad u_{xx} + u_{yy} = 0 \Rightarrow \int_{-\infty}^{+\infty} u_{xx}(x,y) e^{-iwx} dx + \int_{-\infty}^{+\infty} u_{yy}(x,y) e^{-iwx} dx = 0$

$$\Rightarrow (iw)^2 u(w,y) + u_{yy}(w,y) = 0 \Rightarrow -w^2 u(w,y) + u_{yy}(w,y) = 0$$

مساله دیفرانسیل را حل می کنیم:

معادله مشخصه $-w^2+t^2=0 \Rightarrow t=\pm w \Rightarrow u(w,y)=A_1e^{wy}+A_2e^{-wy}$

چون $\lim|u(x,y)|<M \Rightarrow u(w,y)=ce^{-|w|y}\begin{cases} c=A_2 & w>0 \\ c=A_1 & w<0\end{cases}$

$x\to\pm\infty$

$y\to\infty$

$$u(w,0)=c=\int_{-\infty}^{+\infty}F(x)e^{-iwx}dx=\int_{-\infty}^{+\infty}e^{-x2}e^{-iwx}dx$$

$$\Rightarrow u(w,0)=F(e^{-x2})=c=\sqrt{\pi}e^{-\frac{w2}{4}} \Rightarrow u(w,y)=\sqrt{\pi}e^{-\frac{w2}{4}}e^{-|w|y}$$

تبدیل وارون فوریه

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}u(w,y)e^{iwx}dw=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sqrt{\pi}e^{-\frac{w2}{4}}e^{-|w|y}e^{iwx}dw$$

$$\Rightarrow u(x,y)=\frac{1}{2\sqrt{\pi}}\left[\int_{-\infty}^{0}e^{-\frac{w2}{4}}e^{wy}e^{iwx}dw+\int_{0}^{\infty}e^{-\frac{w2}{4}}e^{-wy}e^{iwx}dw\right]$$

$$\Rightarrow u(x,y)=\frac{1}{2\sqrt{\pi}}\left[\int_{-\infty}^{0}e^{w(y+ix-\frac{w}{4})}dw+\int_{0}^{\infty}e^{w(ix-y-\frac{w}{4})}dw\right]$$

با جلسه ی انتگرال بالا $u(x,y)$بدست می آید.

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14) د) $u(x,0)=\frac{1}{x^2+1}$

$$u(x,y)=?\Rightarrow u(w,y)=F(u(x,y))=\sum_{-\infty}^{+\infty}u(x,y)e^{-iwx}dx$$

تبدیل فوریه نسبت به x

$$F(u_{xx})+F(u_{yy})=0\Rightarrow (iw)^2u(w,y)+u_{yy}(w,y)=0$$

$t=\pm w \qquad u(w,y)=A_1e^{wy}+A_2e^{-wy}=ce^{-|w|y}\begin{cases} c=A_2 & w>0 \\ c=A_1 & w<0\end{cases}$

$$u(w,0) = c = F(u(x,0)) = \int_{-\infty}^{+\infty} e^{-iwx}(F(x))dx$$

$$u(w,0) = \int_{-\infty}^{+\infty} \frac{e^{-iwx}}{x^2+1} dx = c$$

$$u(w,y) = ce^{-|w|y} = \left[\int_{-\infty}^{+\infty} \frac{e^{-iwx}}{x^2+1} dx\right] e^{-|w|y}$$

تبديل وارون

$$\longrightarrow$$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(w,y)e^{iwx}dw = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} \frac{e^{-iwx}}{x^2+1} dx\right] e^{-|w|y} e^{iwx} dw$$

هـ) $u(x,0) = \dfrac{x^2}{(x^2+1)^2}$

$$u(x,y) = ? \Rightarrow u(w,y) = F(u(x,y)) = \int_{-\infty}^{+\infty} u(x,y)e^{-iwx}dx$$

تبديل فوريه نسبت x بـه $\Rightarrow F(u_{xx}) + F(u_{yy}) = 0 \Rightarrow (iw)^2 u(w,y) + u_{yy}(w,y) = 0 \Rightarrow$

$$t^2 = w^2 \Rightarrow t = \pm w$$

$$u(w,y) = A_1 e^{wy} + A_2 e^{-wy} = ce^{-|w|y} \begin{cases} c = A_2 & w > 0 \\ c = A_1 & w > 0 \end{cases}$$

$$u(w,0) = c = F(u(x,0)) = \int_{-\infty}^{+\infty} e^{-iwx}(f(x))dx$$

$$u(w,0) = \int_{-\infty}^{+\infty} \frac{e^{-iwx}}{(x^2+1)^2} x^2 dx = c$$

$$u(w,y) = ce^{-|w|y} = \left[\int_{-\infty}^{+\infty} \frac{e^{-iwx}}{(x^2+1)^2} x^2 dx\right] e^{-|w|y}$$

تبديل وارون

$$\Rightarrow u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}u(w,y)e^{iwx}dw=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\frac{e^{-iwx}}{(x^2+1)^2}xdx\right]\times e^{-|w|y}e^{iwx}dw$$

---

15) $\nabla^2 u=0 \qquad -\infty<x<\infty, 0<y<1 \qquad \frac{\delta u}{\delta y}(x,0)=0, u(x,1)e^{-x2}$

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=0 \qquad F\{u(x,y)\}=V(w,y)\Rightarrow -w^2V+\frac{\delta^2 V}{\delta y^2}=0$$

$$\Rightarrow V(w,y)=A(w)e^{wy}+B(w)e^{-wy} \qquad \frac{\delta u}{\delta y}(x,0)=0\Rightarrow \frac{\delta V}{\delta y}(w,0)=0$$

$$\frac{\delta V}{\delta y}=wA(w)e^{wy}-wB(w)e^{-wy}\Rightarrow w(A(w)-B(w))=0\Rightarrow A(w)=B(w)$$

$$\Rightarrow V(w,y)=A(w)\left[e^{wy}+e^{-wy}\right] \qquad u(x,1)=e^{-x2}\Rightarrow V(w,1)=\sqrt{\pi}e^{-\frac{w2}{4}}$$

$$\Rightarrow A(w)\left[e^{w}+e^{-w}\right]=\sqrt{\pi}e^{-\frac{w2}{4}}\Rightarrow A(w)=\frac{\sqrt{\pi}e^{-\frac{w2}{4}}}{\left[e^{w}+e^{-w}\right]}$$

$$\Rightarrow u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sqrt{\pi}e^{-\frac{w2}{4}}}{(e^{w}+e^{-w})}(e^{wy}+e^{-wy})dwe^{iwx}$$

---

16) $\nabla^2 u=0 \qquad -\infty<x<\infty, 0<y/,1 \qquad u(x,0)=\begin{cases}0 & x<0\\ e^{-ax} & x>0\end{cases}, u(x,1)=0$

$$F\{u(x,y)\}=V(w,y)\Rightarrow -w^2V+\frac{\delta^2 u}{\delta y^2}=0\Rightarrow V(w,y)=A(w)e^{wy}+B(w)e^{-wy}$$

$$u(x,0)=\begin{cases}0 & x<0\\ e^{-ax} & x>0\end{cases}$$

$$\Rightarrow V(w,0)=\frac{1}{a+iw}, u(x,1)=0\Rightarrow V(w,1)=0\Rightarrow A(w)e^{w}+B(w)e^{-w}$$

$$V(w,0)=A(w)+B(w)\Rightarrow A(w)+B(w)=\frac{1}{a+iw}\Rightarrow B(w)=\frac{1}{a+iw}-A(w)$$

$$\Rightarrow A(w)e^{w}+\frac{e^{-w}}{a+iw}-A(w)e^{-w}=0\Rightarrow A(w)=(e^{w}-e^{-w})=\frac{-e^{-w}}{a+iw}\Rightarrow A(w)=\frac{e^{-w}}{(e^{-w}+e^{w})(a+iw)}$$

$$\Rightarrow V(w,y)=A(w)(e^{wy}-e^{-wy})+\frac{1}{a+iw}e^{-wy}=\frac{e^{-w}}{(e^{-w}-e^{w})(a-iw)}(e^{wy}-e^{-wy})+\frac{1}{a+iw}e^{-wy}$$

$$\Rightarrow u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}V(w,y)e^{iwx}dw$$

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17) $u_t-u_{xx}+t_u=0 \qquad x>0,t>0 \qquad u(x,0)=xe^{-x}, u_x(0,t)=0$

$$F_c\{u(x,t)\}=V(w,y)\Rightarrow\frac{\delta V}{\delta t}+w^2V+tV=0\Rightarrow\frac{\delta V}{\delta t}+(w^2+t)V=0$$

$$\frac{\delta V}{V}+\left(w^2+t\right)\delta t=0\Rightarrow\ln V+w^2t+\frac{1}{2}t^2=A(w)\Rightarrow\ln V=A(w)-(w^2t+\frac{1}{2}{}^{t}\,2)$$

$$\Rightarrow\ln V=\ln e^{A(w)}-\ln e^{\left(w^2t+\frac{1}{2}t^2\right)}\Rightarrow\ln V=\ln\left[e^{A(w)}\times e^{-(w^2t+\frac{1}{2}t^2)}\right]$$

$$\Rightarrow V=e^{A(w)}\times e^{-(w^2t+\frac{1}{2}t^2)}\Rightarrow V=e^{A(w)-\left(w^2t+\frac{1}{2}t^2\right)} \qquad u(x,0)=xe^{-x}\Rightarrow$$

$$\Rightarrow V(w,0)=\frac{(1-w^2)}{(1+w^2)^2}, V(w,0)=e^{A(w)}\Rightarrow e^{A(w)}=\frac{\left(1-w^2\right)}{\left(1+w^2\right)^2}$$

$$\Rightarrow A(w)=\ln\left[\frac{(1-w^2)}{(1+w^2)^2}\right]\Rightarrow V(w,t)e^{A(w)}\times e^{-(w^2t+\frac{1}{2}t^2)}=\frac{(1-w^2)}{(1+w^2)^2}e^{-(w^2t+\frac{1}{2}t^2)}$$

$$\Rightarrow u(x,t)=\frac{1}{2\pi}\int_{0}^{\infty}\frac{\left(1-w^2\right)}{(1+w^2)^2}e^{-(w^2t+\frac{1}{2}t^2)}Coswxdw$$